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Global and blow-up solutions for quasilinear parabolic equations with a gradient term and nonlinear boundary flux

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Abstract

This work is concerned with positive classical solutions for a quasilinear parabolic equation with a gradient term and nonlinear boundary flux. We find sufficient conditions for the existence of global and blow-up solutions. Moreover, an upper bound for the 'blow-up time', an upper estimate of the 'blow-up rate' and an upper estimate of the global solution are given. Finally, some application examples are presented.

MSC: 35R45; 35K65; 34A12

Keywords: quasilinear parabolic equations; gradient term; boundary flux; blow-up; global solution

1 Introduction

In this paper, we consider the quasilinear parabolic equation with a gradient term

$$\big(g(u)\big)_t = \nabla \cdot \big(a(u)b(x)c(t)\nabla u\big) + f(x,u,q,t) \quad \text{in } D \times (0,T), \tag{1.1}$$

subject to the nonlinear boundary flux and initial conditions

$$\frac{\partial u}{\partial n} = h(x,t)r(u) \quad \text{on } \partial D \times (0,T), \tag{1.2}$$

$$u(x,0) = u_0(x) \quad \text{in } \overline{D}. \tag{1.3}$$

Here $D \subset \mathbb{R}^N$ ($N \geq 1$) is a bounded domain with a smooth boundary ∂D, $\overline{D}$ is the closure of D, $q = |\nabla u|^2$, n is the outer normal vector and T is the maximum existence time of $u(x,t)$. $a(u)b(x)c(t)$, $f(x,u,q,t)$ and $h(x,t)r(u)$ are nonlinear diffusion coefficient, reaction term and boundary flux, respectively. Let $\mathbb{R}^+ = (0,+\infty)$, $\overline{\mathbb{R}^+} = [0,+\infty)$, and suppose that the function $g(s) \in C^2(\mathbb{R}^+)$, $g'(s) > 0$ for any $s > 0$, $a(s) \in C^2(\mathbb{R}^+)$, $b(x) \in C^1(\overline{D})$, $c(t) \in C^1(\mathbb{R}^+)$, $f(x,u,q,t) \in C^1(\overline{D} \times \mathbb{R}^+ \times \overline{\mathbb{R}^+} \times \overline{\mathbb{R}^+})$ is a nonnegative function, $h(x,t) \in C^1(\overline{D} \times (0,T))$, $r(s) \in C^2(\mathbb{R}^+)$ is a positive function, and the positive function $u_0(x) \in C^2(\overline{D})$ satisfies the compatibility conditions. Under these assumptions, the classical parabolic equation theory [1, Section 3] ensures that there exists a unique classical solution $u(x,t)$ to problem (1.1)-(1.3) for some $T > 0$, and the solution is positive over $\overline{D} \times [0,T)$. Moreover, by the regularity theorem [2, Chapter 3], we know $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$.

Equation (1.1) describes the diffusion of concentration of some Newtonian fluids through porous media or the density of some biological species in many physical phenomena and combustion theories (see [3, 4]). The nonlinear Neumann boundary value condition (1.2) can be physically interpreted as the nonlinear radial law (see, *e.g.*, [5, 6]).

In recent years the questions like blow-up and global solvability for nonlinear evolution equations have been investigated extensively by many authors. In particular, for the parabolic equations with a gradient term, we refer to [7–12] *etc.* For example, Souplet and Weissler [7] studied the semilinear parabolic equation

$$u_t = \Delta u + f(u, \nabla u) \quad \text{in } D \times (0, T),$$

subject to the homogeneous Dirichlet boundary condition. By using the comparison principle and constructing a self-similar lower solution, they obtained sufficient conditions for global existence and blow-up solutions. Andreu [8] used a similar method to study the quasilinear parabolic equation

$$u_t = \Delta u^m + f\big(u, \nabla u^m\big) \quad \text{in } D \times (0, T).$$

Chen [9] considered the following semilinear parabolic equation:

$$u_t = \Delta u + f(u) + g(u)|\nabla u|^2 \quad \text{in } D \times (0, T),$$

with the homogeneous Dirichlet boundary condition. By estimating the integral of ratio of one solution to the other, the author proved both global existence and blow-up results. Then he used the same method to study a more generalized equation with a gradient term, see [10].

For the nonlinear parabolic equations with Neumann boundary conditions, Lair and Oxley [11] considered the quasilinear parabolic equation without a gradient term

$$u_t = \nabla \cdot \big(a(u)\nabla u\big) + f(u) \quad \text{in } D \times (0, T),$$

subject to the homogeneous Neumann boundary conditions, and they obtained the necessary and sufficient conditions for the global existence and blow-up solution by the approximation method. Recently, Ding and Gao [12] investigated an initial boundary value problem of the quasilinear parabolic equation with a gradient term

$$\big(g(u)\big)_t = \Delta u + f\big(x, u, |\nabla u|^2, t\big) \quad \text{in } D \times (0, T),$$

subject to boundary flux $\frac{\partial u}{\partial n} = r(u)$, and they obtained sufficient conditions for the global existence and blow-up solution, the upper estimate of global solution and blow-up time.

Motivated by the above works, we construct an appropriate auxiliary function and use the Hopf maximum principle to study problem (1.1)-(1.3). The aim of this paper is to obtain sufficient conditions for the existence of blow-up and global solution, an upper bound for the 'blow-up time', an upper estimate of the 'blow-up rate' and an upper estimate of the global solution and then to give some examples.

2 Main results and proof

We now state and prove the main results of this paper. Firstly, we give sufficient conditions of the existence of a blow-up solution of problem (1.1)-(1.3).

Theorem 1 *Let $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ be a solution of problem* (1.1)-(1.3). *Assume that the following conditions hold*:

(1) *For any $(x,s,q,t) \in \overline{D} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$,*

$$a(s) > 0, \qquad b(x) > 0, \qquad c(t) > 0, \qquad r(s) > 0, \qquad h(x,t) \geq 0; \tag{2.1}$$

(2) *For any $(x,s,q,t) \in \overline{D} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$,*

$$\begin{aligned} & a'(s) \geq 0, \qquad h_t(x,t) \geq 0, \qquad f_q \geq 0, \qquad \left(\frac{a(s)}{g'(s)}\right)' \geq 0, \qquad r'(s) \geq \frac{a'(s)}{a(s)} r(s), \\ & r''(s) \geq \frac{a'(s)}{a(s)} r'(s), \end{aligned} \tag{2.2}$$

$$\begin{aligned} & c'(t) \geq 0, \qquad g'(s) > 0, \qquad f_t(x,s,q,t) \geq \frac{c'(t)}{c(t)} f(x,s,q,t), \\ & f_s(x,s,q,t) \geq \frac{r'(s)}{r(s)} f(x,s,q,t); \end{aligned} \tag{2.3}$$

(3) *For any $x \in \{x \mid f(x,u_0,q_0,0) = 0, x \in \overline{D}\}$,*

$$\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big) \geq 0; \tag{2.4}$$

(4) *The constant*

$$\beta = \min_{D_1} \left\{ \frac{a(u_0)}{g'(u_0)r(u_0)} \big[\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big) + f(x,u_0,q_0,0)\big] \right\} > 0, \tag{2.5}$$

where $D_1 = \{x \mid f(x,u_0,q_0,0) \neq 0, x \in \overline{D}\} \neq \phi$, $q_0 = |\nabla u_0|^2$;

(5) *The integration*

$$\int_{M_0}^{+\infty} \frac{a(s)}{r(s)}\, ds < +\infty, \qquad \text{where } M_0 = \max_{\overline{D}} u_0(x); \tag{2.6}$$

then the solution $u(x,t)$ of system (1.1)-(1.3) *must blow up in finite time T and*

$$T \leq \frac{1}{\beta} \int_{M_0}^{+\infty} \frac{a(s)}{r(s)}\, ds, \tag{2.7}$$

$$u(x,t) \leq \Phi^{-1}\big(\beta(T-t)\big), \tag{2.8}$$

where $\Phi(z) = \int_z^{+\infty} \frac{a(s)}{r(s)}\, ds$, $z > 0$, and Φ^{-1} is the inverse function of Φ.

Proof Consider the auxiliary function

$$\Psi = -\frac{1}{r(u)} u_t + \beta \frac{1}{a(u)}. \tag{2.9}$$

We find that

$$\nabla\Psi = \frac{r'}{r^2}u_t\nabla u - \frac{1}{r}\nabla u_t - \beta\frac{a'}{a^2}\nabla u, \tag{2.10}$$

$$\begin{aligned}\Delta\Psi &= \left(\frac{r''}{r^2} - 2\frac{(r')^2}{r^3}\right)qu_t + 2\frac{r'}{r^2}\nabla u\cdot\nabla u_t + \frac{r'}{r^2}u_t\Delta u - \frac{1}{r}\Delta u_t \\ &\quad - \beta\left(\frac{a''}{a^2} - 2\frac{(a')^2}{a^3}\right)q - \beta\frac{(a')}{a^2}\Delta u,\end{aligned} \tag{2.11}$$

and

$$\begin{aligned}\Psi_t &= \frac{r'}{r^2}(u_t)^2 - \frac{1}{r}(u_t)_t - \beta\frac{(a')}{a^2}u_t \\ &= \frac{r'}{r^2}(u_t)^2 - \frac{1}{r}\left[\frac{1}{g'}\big(abc\Delta u + a'bcq + ac\nabla b\cdot\nabla u + f\big)\right]_t - \beta\frac{(a')}{a^2}u_t \\ &= \frac{r'}{r^2}(u_t)^2 - \beta\frac{(a')}{a^2}u_t - \frac{1}{g'r}\big(a'bcu_t\Delta u + abc'\Delta u + abc\Delta u_t + a''bcqu_t \\ &\quad + a'bc'q + 2a'bc\nabla u\cdot\nabla u_t \\ &\quad + a'cu_t\nabla b\cdot\nabla u + ac'\nabla b\cdot\nabla u + ac\nabla b\cdot\nabla u_t + 2f_q\nabla u\cdot\nabla u_t + f_t + f_u u_t\big) \\ &\quad + \frac{g''}{(g')^2 r}\big(abc\Delta u + a'bcq + ac\nabla b\cdot\nabla u + f\big)u_t.\end{aligned} \tag{2.12}$$

Hence, from (2.11) and (2.12) we have

$$\begin{aligned}&\frac{abc}{g'}\Delta\Psi - \Psi_t \\ &\quad = \left(\frac{abc}{g'}\frac{r''}{r^2} - 2\frac{abc}{g'}\frac{(r')^2}{r^3} + \frac{a''bc}{g'}\frac{1}{r} - \frac{a'bc}{r}\frac{g''}{(g')^2}\right)qu_t \\ &\qquad + \left(2\frac{abc}{g'}\frac{r'}{r^2} + 2\frac{abc}{g'}\frac{1}{r} + 2\frac{f_q}{g'}\frac{1}{r}\right)\nabla u\cdot\nabla u_t \\ &\qquad + \left(\frac{abc}{g'}\frac{r'}{r^2} + \frac{abc}{g'}\frac{1}{r} - \frac{abc}{r}\frac{g''}{(g')^2}\right)u_t\Delta u + \left(\frac{a'bc'}{g'}\frac{1}{r} - \beta\frac{abc}{g'}\frac{a''}{a^2} + 2\beta\frac{abc}{g'}\frac{(a')^2}{a^3}\right)q \\ &\qquad + \left(\frac{a'bc'}{g'}\frac{1}{r} - \beta\frac{abc}{g'}\frac{a'}{a^2}\right)\Delta u - \frac{r'}{r^2}(u_t)^2 + \beta\frac{a'}{a^2}u_t + \frac{a'c}{g'r}u_t\nabla b\cdot\nabla u + \frac{ac'}{g'r}\nabla b\cdot\nabla u \\ &\qquad + \frac{f_t}{g'r} + \frac{f_u}{g'r}u_t - \frac{ac}{r}\frac{g''}{(g')^2}u_t\nabla b\cdot\nabla u - \frac{f}{r}\frac{g''}{(g')^2}u_t.\end{aligned} \tag{2.13}$$

Using (2.10) leads to

$$\nabla u_t = -r\nabla\Psi - \beta\frac{a'r}{a^2}\nabla u + \frac{r'}{r}u_t\nabla u. \tag{2.14}$$

Now substituting (2.14) into (2.13) yields

$$\begin{aligned}&\frac{abc}{g'}\Delta\Psi + \left(\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u + 2\frac{bc}{g'}\frac{(ar)'}{r}\nabla u\right)\nabla\Psi - \Psi_t \\ &\quad = \left(\frac{abc}{g'}\frac{r''}{r^2} + \frac{a''bc}{g'}\frac{1}{r} + 2\frac{a'bc}{g'}\frac{r'}{r^2} + 2\frac{f_q}{g'}\frac{r'}{r} - \frac{a'bc}{r}\frac{g''}{(g')^2}\right)qu_t\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{abc'}{g'}\frac{1}{r}-\beta\frac{abc}{g'}\frac{a'}{a^2}\right)\Delta u\\
&+\left(\frac{abc}{g'}\frac{r'}{r^2}+\frac{abc}{g'}\frac{1}{r}-\frac{abc}{r}\frac{g''}{(g')^2}\right)u_t\Delta u+\left(\frac{a'c}{g'r}+\frac{ac}{g'}\frac{r'}{r^2}-\frac{ac}{r}\frac{g''}{(g')^2}\right)u_t\nabla b\cdot\nabla u\\
&+\left(\frac{a'bc'}{g'}\frac{1}{r}-\beta\frac{abc}{g'}\frac{a''}{a^2}-2\beta\frac{abc}{g'}\frac{a'}{a^2}\frac{r'}{r}-2\beta\frac{f_q}{g'}\frac{a'r}{a}\right)q+\left(\beta\frac{a'}{a^2}+\frac{f_u}{g'r}-\frac{f}{r}\frac{g''}{(g')^2}\right)u_t\\
&-\frac{r'}{r^2}(u_t)^2+\left(\frac{ac'}{g'r}-\beta\frac{ac}{g'}\frac{a'}{a^2}\right)\nabla b\cdot\nabla u+\frac{f_t}{g'r}.
\end{aligned}
\tag{2.15}
$$

In fact, from (1.1) we see that

$$
\Delta u=\frac{1}{abc}\bigl(g'u_t-a'bcq-ac\nabla b\cdot\nabla u-f\bigr).
\tag{2.16}
$$

Thus combining (2.15) and (2.16), we arrive at

$$
\begin{aligned}
&\frac{abc}{g'}\Delta\Psi+\left(\frac{ac}{g'}\nabla b+2\frac{f_q}{g'}\nabla u+2\frac{bc}{g'}\frac{(ar)'}{r}\nabla u\right)\nabla\Psi-\Psi_t\\
&\quad=\left(\frac{abc}{g'}\frac{r''}{r^2}+\frac{a''bc}{g'}\frac{1}{r}+\frac{a'bc}{g'}\frac{r'}{r^2}+2\frac{f_q}{g'}\frac{r'}{r}-\frac{(a')^2bc}{ag'}\frac{1}{r}\right)qu_t\\
&\qquad+\left(\beta\frac{(a')^2bc'}{a^2g'}-\beta\frac{a''bc}{ag'}-2\beta\frac{a'bc}{ag'}\frac{1}{r}-2\beta\frac{a'r}{a^2}\frac{f_q}{g'}\right)q\\
&\qquad+\left(\frac{c'}{c}\frac{1}{r}-\frac{f}{g'}\frac{r'}{r^2}-\frac{a'}{a}\frac{f}{g'}\frac{1}{r}+\frac{f_u}{g'r}\right)u_t+\frac{f_t}{g'r}-\frac{c'}{c}\frac{f}{g'r}\\
&\qquad+\beta\frac{a'}{a^2}\frac{f}{g'}+\left(\frac{a'}{a}\frac{1}{r}-\frac{1}{r}\frac{g''}{(g')^2}\right)(u_t)^2.
\end{aligned}
\tag{2.17}
$$

In view of (2.9), we have

$$
u_t=-r\Psi+\beta\frac{r}{a}.
\tag{2.18}
$$

If we substitute (2.18) into (2.17), then it is easy to obtain

$$
\begin{aligned}
&\frac{abc}{g'}\Delta\Psi+\left(\frac{ac}{g'}\nabla b+2\frac{f_q}{g'}\nabla u+2\frac{bc}{g'}\frac{(ar)'}{r}\nabla u\right)\nabla\Psi\\
&\qquad+\left\{\left[\left(\frac{a'r}{a}\right)'+r''\right]q\frac{abc}{g'r}+2\frac{f_q}{g'}\frac{r'}{r}q+\frac{ar}{g'}\left(\frac{f}{ar}\right)_u+\frac{c'}{c}\right\}\Psi-\Psi_t\\
&\quad=\beta\frac{bc}{g'}\left(\frac{r''}{r}-\frac{a'}{a}\frac{r'}{r}\right)q+2\beta\frac{f_q}{g'}\left(\frac{r'}{a}-\frac{a'r}{a^2}\right)q+\beta\frac{c'}{ac}+\left(\frac{a'}{a}\frac{1}{r}-\frac{1}{r}\frac{g''}{g'}\right)(u_t)^2\\
&\qquad+\frac{1}{g'r}\left(f_t-\frac{c'}{c}f\right)+\beta\frac{1}{ag'}\left(f_u-f\frac{r'}{r}\right)\\
&\quad=\beta\frac{abc}{g'r}\left(\frac{r'}{a}\right)'q+2\beta\frac{f_q}{g'}\left(\frac{r}{a}\right)'q+\beta\frac{c'}{ac}+\frac{g'}{ar}\left(\frac{a}{g'}\right)'(u_t)^2\\
&\qquad+\frac{c}{g'r}\left(\frac{f}{c}\right)_t+\beta\frac{r}{ag'}\left(\frac{f}{r}\right)_u.
\end{aligned}
\tag{2.19}
$$

From assumptions (2.1)-(2.3), it follows that the right-hand side of (2.19) is nonnegative, *i.e.*,

$$\frac{abc}{g'}\Delta\Psi+\left(\frac{ac}{g'}\nabla b+2\frac{f_q}{g'}\nabla u+2\frac{bc}{g'}\frac{(ar)'}{r}\nabla u\right)\nabla\Psi$$
$$+\left\{\left[\left(\frac{a'r}{a}\right)'+r''\right]q\frac{abc}{g'r}+2\frac{f_q}{g'}\frac{r'}{r}q+\frac{ar}{g'}\left(\frac{f}{ar}\right)_u+\frac{c'}{c}\right\}\Psi-\Psi_t\geq 0. \tag{2.20}$$

Then from (2.4) and (2.5) we have

$$\max_{\overline{D}}\Psi(x,0)=\max_{\overline{D}}\left\{-\frac{1}{g'(u_0)r(u_0)}\left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right)+f(x,u_0,q_0,0)\right]+\beta\frac{1}{a(u_0)}\right\}$$
$$\leq 0. \tag{2.21}$$

And as we can see, an explicit calculation

$$\frac{\partial\Psi}{\partial n}=\frac{r'}{r^2}u_t\frac{\partial u}{\partial n}-\frac{1}{r}\frac{\partial u_t}{\partial n}-\beta\frac{a'}{a^2}\frac{\partial u}{\partial n}=\frac{r'}{r^2}hu_t-\frac{1}{r}(hr)_t-\beta\frac{a'}{a^2}hr$$
$$=\frac{r'}{r^2}hu_t-h_t-\frac{r'}{r}hu_t-\beta\frac{a'}{a^2}hr=-h_t-\beta\frac{a'}{a^2}hr\leq 0 \tag{2.22}$$

holds on $\partial D\times(0,T)$. Thus, by combining (2.20)-(2.22) and using the Hopf maximum principle, we find that the maximum of Ψ on $\partial D\times(0,T)$ is 0, *i.e.*,

$$\Psi\leq 0\quad\text{on } \partial D\times(0,T),$$

and by (2.9), it gives

$$\frac{a(u)}{r(u)}u_t\geq\beta. \tag{2.23}$$

Integrating (2.23) over $[0,t]$ at the point $x_0\in\overline{D}$, where $u_0(x_0)=M_0$, yields

$$\frac{1}{\beta}\int_{M_0}^{u(x_0,t)}\frac{a(s)}{r(s)}\,ds\geq t. \tag{2.24}$$

This together with assumption (2.6) shows that $u(x,t)$ must blow up in finite time T; moreover,

$$T\leq\frac{1}{\beta}\int_{M_0}^{+\infty}\frac{a(s)}{r(s)}\,ds. \tag{2.25}$$

For each fixed x, integrating inequality (2.23) over $[t,s]$ $(0<t<s<T)$ leads to

$$\Phi\big(u(x,t)\big)\geq\Phi\big(u(x,t)\big)-\Phi\big(u(x,s)\big)=\int_{u(x,t)}^{u(x,s)}\frac{a(s)}{r(s)}\,ds\geq\beta(s-t).$$

If we let $s\to T$, then formally

$$\Phi\big(u(x,t)\big)\geq\beta(T-t),$$

therefore

$$u(x,t) \le \Phi^{-1}\big(\beta(T-t)\big).$$

The proof is completed. □

The result on the global solution is stated as Theorem 2 below.

Theorem 2 *Let $u \in C^3(D \times (0,T)) \cap C^2(\overline{D} \times (0,T))$ be a solution of problem* (1.1)-(1.3). *Assume that the following conditions hold*:

(1) *For any $(x,s,q,t) \in \overline{D} \times \mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}^+$,*

$$a(s) > 0, \qquad b(x) > 0, \qquad c(t) > 0, \qquad r(s) > 0, \qquad h(x,t) \ge 0; \tag{2.26}$$

(2) *For any $(x,s,q,t) \in \overline{D} \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$,*

$$\begin{gathered} a'(s) \le 0, \qquad h_t(x,t) \le 0, \qquad f_q \le 0, \qquad \left(\frac{a(s)}{g'(s)}\right)' \le 0, \\ r'(s) \ge \frac{a'(s)}{a(s)} r(s), \qquad r''(s) \le \frac{a'(s)}{a(s)} r'(s), \end{gathered} \tag{2.27}$$

$$\begin{gathered} c'(t) \le 0, \qquad g'(s) > 0, \qquad f_t(x,s,q,t) \le \frac{c'(t)}{c(t)} f(x,s,q,t), \\ f_s(x,s,q,t) \le \frac{r'(s)}{r(s)} f(x,s,q,t); \end{gathered} \tag{2.28}$$

(3) *For any $x \in \{x \mid f(x,u_0,q_0,0) = 0, x \in \overline{D}\}$,*

$$\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big) \ge 0; \tag{2.29}$$

(4) *The constant*

$$\alpha = \max_{D_1}\left\{\frac{a(u_0)}{g'(u_0)r(u_0)}\big[\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big) + f(x,u_0,q_0,0)\big]\right\} > 0, \tag{2.30}$$

where $D_1 = \{x \mid f(x,u_0,q_0,0) = 0, x \in \overline{D}\} \neq \phi$, $q_0 = |\nabla u_0|^2$;

(5) *The integration*

$$\int_{m_0}^{+\infty} \frac{a(s)}{r(s)}\,ds < +\infty, \quad \text{where } m_0 = \min_{\overline{D}} u_0(x); \tag{2.31}$$

then the solution $u(x,t)$ of system (1.1)-(1.3) *must be a global solution and*

$$u(x,t) \le \Psi^{-1}\big(\alpha t + \Psi\big(u_0(x)\big)\big), \tag{2.32}$$

where $\Psi(z) = \int_{m_0}^{z} \frac{a(s)}{r(s)}\,ds$, $z > 0$, and Ψ^{-1} is the inverse function of Ψ.

Proof Consider the auxiliary function

$$\Phi = -\frac{1}{r(u)} u_t + \alpha \frac{1}{a(u)}. \tag{2.33}$$

We first replace Ψ and β in (2.20) with Φ and α, respectively, and under assumptions (2.26)-(2.28), we get

$$\frac{abc}{g'}\Delta\Phi - \left(\frac{ac}{g'}\nabla b + 2\frac{f_q}{g'}\nabla u + 2\frac{bc}{g'}\frac{(ar)'}{r}\nabla u\right)\nabla\Phi + \left\{\left[\left(\frac{a'r}{a}\right)' + r''\right]q\frac{abc}{g'r} + 2\frac{f_q}{g'}\frac{r'}{r}q + \frac{ar}{g'}\left(\frac{f}{ar}\right)_u + \frac{c'}{c}\right\}\Phi - \Phi_t \le 0. \tag{2.34}$$

In fact, from (2.29) and (2.30) we can see that

$$\min_{\overline{D}} \Phi(x,0) = \min_{\overline{D}}\left\{-\frac{1}{g'(u_0)r(u_0)}\left[\nabla\big(a(u_0)b(x)c(0)\nabla u_0\big) + f(x,u_0,q_0,0)\right] + \alpha\frac{1}{a(u_0)}\right\} \ge 0. \tag{2.35}$$

Also, on $\partial D \times (0,T)$, it gives

$$\frac{\partial\Phi}{\partial n} = -h_t - \alpha\frac{a'}{a^2}hr \ge 0. \tag{2.36}$$

By combining (2.34)-(2.36) and using the Hopf maximum principle, we find that the minimum of Φ on $\partial D \times (0,T)$ is 0, *i.e.*,

$$\Phi \ge 0 \quad \text{in } \partial D \times (0,T),$$

and by (2.33), we can see that

$$\frac{a(u)}{r(u)}u_t \le \alpha. \tag{2.37}$$

For each fixed x, integrating (2.37) over $[0,t]$ yields

$$\frac{1}{\alpha}\int_{u_0(x)}^{u(x,t)} \frac{a(s)}{r(s)}\,ds \le t. \tag{2.38}$$

This together with assumption (2.31) shows that $u(x,t)$ must be a global solution; moreover,

$$\Psi\big(u(x,t)\big) - \Psi\big(u_0(x)\big) = \int_{u_0(x)}^{u(x,t)} \frac{a(s)}{r(s)}\,ds \le \alpha t,$$

therefore

$$u(x,t) \le \Psi^{-1}\big(\alpha t + \Psi\big(u_0(x)\big)\big).$$

The proof is completed. □

3 Applications

In what follows, we present several examples to demonstrate the applications of Theorems 1 and 2.

Example 1 Let u be a solution of

$$\begin{aligned}
&(e^{2u})_t = \nabla\cdot\left(e^{3u}\left(1+\sum_{i=1}^{3}x_i^2\right)e^t\nabla u\right)+\left(1+\sum_{i=1}^{3}x_i^2\right)e^{4u}qe^t \quad \text{in } D\times(0,T),\\
&\frac{\partial u}{\partial n} = 2\left(1+t\sum_{i=1}^{3}x_i^4\right)e^{4u} \quad \text{on } \partial D\times(0,T),\\
&u(x,0)=u_0(x)=1+e^4\sum_{i=1}^{3}x_i^2 \quad \text{in } \overline{D},
\end{aligned}$$

where $D=\{x=(x_1,x_2,x_3)\mid\sum_{i=1}^{3}x_i^2<1\}$, then we have

$$\begin{aligned}
&g(u)=e^{2u},\qquad a(u)=e^{3u},\qquad b(x)=1+\sum_{i=1}^{3}x_i^2,\qquad c(t)=e^t,\\
&f(x,u,q,t)=\left(1+\sum_{i=1}^{3}x_i^2\right)e^{4u}qe^t,\qquad h(x,t)=2\left(1+t\sum_{i=1}^{3}x_i^4\right),\qquad r(u)=e^{4u}.
\end{aligned}$$

It is easy to verify that (2.1)-(2.4) hold. By (2.5), we find

$$\begin{aligned}
\beta &= \min_{D_1}\left\{\frac{a(u_0)}{g'(u_0)r(u_0)}\left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right)+f(x,u_0,q_0,0)\right]\right\}\\
&= \min_{1\le u_0<1+e^4}\left\{\frac{1}{2}\left[3u_0|\nabla u_0|^2+|\nabla u_0|^2+u_0\Delta u_0+e^{u_0}u_0|\nabla u_0|^2\right]\right\}=3e^4.
\end{aligned}$$

It follows from Theorem 1 that $u(x,t)$ must blow up in finite time T and

$$T\le\frac{1}{\beta}\int_{M_0}^{+\infty}\frac{a(s)}{r(s)}\,ds=\frac{1}{\beta}\int_{2}^{+\infty}\frac{e^{3s}}{e^{4s}}\,ds=\frac{1}{3}e^{-6},$$

and

$$u(x,t)\le\Phi^{-1}\left(\beta(T-t)\right)=\ln\left[\frac{1}{3e^4}(T-t)^{-1}\right].$$

Example 2 Let u be a solution of

$$\begin{aligned}
&(u\sqrt{u})_t=\nabla\cdot\left(\frac{1}{\sqrt{u}}\left(1+\sum_{i=1}^{3}x_i^2\right)\frac{1}{1+t}\nabla u\right)+\left(1+\sum_{i=1}^{3}x_i^2\right)\frac{1-q}{1+t}\sqrt{u} \quad \text{in } D\times(0,T),\\
&\frac{\partial u}{\partial n}=\sqrt{2}\left(1+t\sum_{i=1}^{3}x_i^4\right)^{-1}\sqrt{u} \quad \text{in } \partial D\times(0,T),\\
&u(x,0)=u_0(x)=1+\sum_{i=1}^{3}x_i^2 \quad \text{in } \overline{D},
\end{aligned}$$

where $D = \{x = (x_1, x_2, x_3) \mid \sum_{i=1}^{3} x_i^2 < 1\}$, then we have

$$g(u) = u\sqrt{u}, \qquad a(u) = \frac{1}{\sqrt{u}}, \qquad b(x) = \left(1 + \sum_{i=1}^{3} x_i^2\right), \qquad c(t) = \frac{1}{1+t},$$
$$f(x,u,q,t) = \left(1 + \sum_{i=1}^{3} x_i^2\right)\frac{1-q}{1+t}\sqrt{u}, \qquad h(x,t) = \sqrt{2}\left(1 + t\sum_{i=1}^{3} x_i^4\right)^{-1}, \qquad r(u) = \sqrt{u}.$$

It is easy to verify that (2.26)-(2.29) hold. By (2.30), we find

$$\begin{aligned}\alpha &= \max_{D_1}\left\{\frac{a(u_0)}{g'(u_0)r(u_0)}\left[\nabla\left(a(u_0)b(x)c(0)\nabla u_0\right) + f(x,u_0,q_0,0)\right]\right\} \\ &= \max_{1\le u_0<2}\left\{\frac{1}{3}\left[-u_0^{-2}|\nabla u_0|^2 + 2u_0^{-1}\Delta u_0 + 2\left(1 - |\nabla u_0|^{-2}\right)\right]\right\} = \frac{14}{3}.\end{aligned}$$

It follows from Theorem 2 that $u(x,t)$ must be a global solution and

$$u(x,t) \le \Psi^{-1}\left(\alpha t + \Psi\left(u_0(x)\right)\right) = \exp(\alpha t + \ln u_0) = u_0 \exp\left(\frac{14}{3}t\right).$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the manuscript and read and approved the final manuscript.

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