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The first Seiffert mean is strictly (G, A)-super-stabilizable

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Abstract
The concept of strictly super-stabilizability for bivariate means has been defined recently by Raïsoulli and Sándor (J. Inequal. Appl. 2014:28, 2014). We answer into affirmative to an open question posed in that paper, namely: Prove or disprove that the first Seiffert mean P is strictly (G, A)-super-stabilizable. We use series expansions of the functions involved and reduce the main inequality to three auxiliary ones. The computations are performed with the aid of the computer algebra systems *Maple* and *Maxima*. The method is general and can be adapted to other problems related to sub- or super-stabilizability.
MSC: 26E60

Keywords: means; stable means; strictly super-stabilizable means

Introduction

A *bivariate mean* is a map $m : (0,\infty)^2 \to \mathbb{R}$ satisfying the following statement:

$$\forall a,b > 0, \quad \min(a,b) \le m(a,b) \le \max(a,b).$$

Obviously $m(a,a) = a$ for each $a > 0$. The maps $(a,b) \mapsto \min(a,b)$ and $(a,b) \mapsto \max(a,b)$ are means, and they are called the *trivial means*.

A mean m is *symmetric* if $m(a,b) = m(b,a)$ for all $a,b > 0$, and *monotone* if $(a,b) \mapsto m(a,b)$ is increasing in a and in b, that is, if $a_1 \le a_2$ (respectively $b_1 \le b_2$) then $m(a_1,b) \le m(a_2,b)$ (respectively $m(a,b_1) \le m(a,b_2)$). For more details as regards monotone means, see [1].

For two means m_1 and m_2 we write $m_1 \le m_2$ if and only if $m_1(a,b) \le m_2(a,b)$ for every $a,b > 0$, and $m_1 < m_2$ if and only if $m_1(a,b) < m_2(a,b)$ for all $a,b > 0$ with $a \ne b$. Two means m_1 and m_2 are *comparable* if $m_1 \le m_2$ or $m_2 \le m_1$, and we say that m is between two comparable means m_1 and m_2 if $\min(m_1,m_2) \le m \le \max(m_1,m_2)$. If the above inequalities are strict then we say that m is *strictly between* m_1 and m_2.

Some standard examples of means are given in the following (see [2] and the references therein):

$$A := A(a,b) = \frac{a+b}{2}; \qquad G := G(a,b) = \sqrt{ab}; \qquad H := H(a,b) = \frac{2ab}{a+b};$$

$$L := L(a,b) = \frac{b-a}{\ln b - \ln a}, \qquad L(a,a) = a; \qquad I := I(a,b) = e^{-1}\left(\frac{b^b}{a^a}\right)^{1/(b-a)},$$

$$I(a,a) = a;$$

$$P := P(a,b) = \frac{b-a}{4\arctan\sqrt{b/a} - \pi} = \frac{b-a}{2\arcsin\frac{b-a}{b+a}},$$

$$P(a,a) = a,$$

and are called the arithmetic, geometric, harmonic, logarithmic, identric means, respectively, the first Seiffert mean.

The above means are strictly comparable, namely

$$\min < H < G < L < P < I < A < \max.$$

The next section presents some definitions and preliminary results, and the last section contains the main result. Its proof is based on some heavy computations, and a computer algebra system may be very helpful.

We have used *Maple* and *Maxima*, which already offered good results in proving inequalities for means (see, for example, [3]). Note that all the symbolic computations are exact, because only polynomials with rational coefficients are involved. We would like to point out that the method used in this paper is easily adaptable to other 'stiff' inequalities involving real analytic functions, if they contain subexpressions with algebraic derivatives.

In particular, during the proof, we needed the Sturm sequence associated to a univariate polynomial, say p, in order to find the number of roots in intervals $(c,d]$. This can be obtained in *Maple* by

$$\texttt{sturm}\big(\texttt{sturmseq}(p,c,d)\big)$$

or in *Maxima* by

$$\texttt{nroot}(p,c,d),$$

both making use of exact (rational) arithmetic.

Definitions and preliminary results

At first we define the resultant mean-map of three means as in [4], where the properties of the resultant mean-map are studied.

Definition 1 Let m_1, m_2, and m_3 be three given symmetric means. For all $a,b > 0$, define the *resultant mean-map* of m_1, m_2, and m_3 as

$$\mathcal{R}(m_1,m_2,m_3)(a,b) = m_1\big(m_2\big(a,m_3(a,b)\big),m_2\big(m_3(a,b),b\big)\big).$$

Example 2 For $m_1 = G$, $m_3 = A$ and $m_2 = P$ we get

$$\mathcal{R}(G,P,A)(a,b) = \frac{|b-a|}{4}\left(\arcsin\frac{b-a}{3a+b}\arcsin\frac{b-a}{a+3b}\right)^{-1/2}. \qquad (1)$$

Definition 3 A symmetric mean m is said to be

(a) *stable* if $\mathcal{R}(m,m,m) = m$;

(b) *stabilizable* if there exist two nontrivial stable means m_1 and m_2 satisfying the relation $\mathcal{R}(m_1, m, m_2) = m$. We then say that m is (m_1, m_2)-stabilizable.

A study about the stability and stabilizability of the standard means was presented in [4]. For example, the arithmetic, geometric, and harmonic means A, G, and H are stable. The logarithmic mean L is (H, A)-stabilizable and (A, G)-stabilizable, and the identric mean I is (G, A)-stabilizable.

The next definitions were formulated in [5].

Definition 4 Let m_1, m_2 be two nontrivial stable comparable means. A mean m is called:

(a) (m_1, m_2)-*sub-stabilizable* if $\mathcal{R}(m_1, m, m_2) \le m$ and m is between m_1 and m_2;

(b) (m_1, m_2)-*super-stabilizable* if $m \le \mathcal{R}(m_1, m, m_2)$ and m is between m_1 and m_2.

This definition extends that of stabilizability, in the sense that a mean m is (m_1, m_2)-stabilizable if and only if (a) and (b) hold.

Definition 5 Let m_1, m_2 be two nontrivial stable comparable means. A mean m is called:

(a) *strictly* (m_1, m_2)-*sub-stabilizable* if $\mathcal{R}(m_1, m, m_2) < m$ and m is strictly between m_1 and m_2;

(b) *strictly* (m_1, m_2)-*super-stabilizable* if $m < \mathcal{R}(m_1, m, m_2)$ and m is strictly between m_1 and m_2.

Example 6 [5] The geometric mean G is (G, A)-super-stabilizable (but not strictly), and A is (G, A)-sub-stabilizable.

The logarithmic mean L is strictly (G, A)-super-stabilizable and strictly (A, H)-sub-stabilizable. The identric mean I is strictly (A, G)-sub-stabilizable.

It is not known if the first Seiffert mean P is stabilizable or not. Several inequalities related to Seiffert means can be found in [6–10] and the references therein.

In [5] it was proved that the first Seiffert mean P is strictly (A, G)-sub-stabilizable. An open problem was proposed there, namely: prove or disprove that the first Seiffert mean P is strictly (G, A)-super-stabilizable.

In what follows we shall prove that indeed the first Seiffert mean P is strictly (G, A)-super-stabilizable.

Main result

It is well known that $G < P < A$ and both G and A are stable. We have to prove that $P < \mathcal{R}(G, P, A)$, which, using (1), is equivalent with

$$\frac{b-a}{2\arcsin\frac{b-a}{b+a}} < \frac{|b-a|}{4}\left(\arcsin\frac{b-a}{3a+b}\arcsin\frac{b-a}{a+3b}\right)^{-1/2},$$

or

$$4\arcsin\frac{b-a}{3a+b}\arcsin\frac{b-a}{a+3b} < \left(\arcsin\frac{b-a}{b+a}\right)^2, \tag{2}$$

for all $a, b > 0$ with $a \neq b$. Without restricting the generality, we may consider that $b > a$ and after the substitution $t = (b - a)/(b + a)$ we reduce the problem to

$$(\arcsin t)^2 > 4 \arcsin \frac{t}{2 - t} \arcsin \frac{t}{2 + t} \tag{3}$$

for all $0 < t < 1$.

Theorem 7 *The first Seiffert mean P is strictly* (G, A)*-super-stabilizable.*

Proof We have to prove that (3) holds for $0 < t < 1$. To this aim we denote, for $0 \leq t \leq 1$,

$$\alpha(t) = (\arcsin t)^2, \qquad \beta(t) = \arcsin \frac{t}{2 + t}, \qquad \gamma(t) = \arcsin \frac{t}{2 - t}, \tag{4}$$

and we shall prove that

$$\alpha(t) > 4\beta(t)\gamma(t) \tag{5}$$

for all $0 < t < 1$. For $r = 0.995$, the inequality (5) is true on $[r, 1)$ because the functions α, β, γ are all increasing and for $t \geq r$

$$\alpha(t) - 4\beta(t)\gamma(t) \geq \alpha(r) - 4\beta(1)\gamma(1) = 0.027\ldots > 0. \tag{6}$$

In order to prove that (5) is true also on $(0, r)$ we shall use series expansions up to 20th degree. We obtain

$$\beta(t) = p_\beta(t) + O\bigl(t^{20}\bigr), \qquad \gamma(t) = p_\gamma(t) + O\bigl(t^{20}\bigr),$$

where

$$\begin{aligned} p_\beta(t) = {}& \frac{1}{2}t - \frac{1}{4}t^2 + \frac{7}{48}t^3 - \frac{3}{32}t^4 + \frac{83}{1{,}280}t^5 - \frac{73}{1{,}536}t^6 + \frac{523}{14{,}336}t^7 - \frac{119}{4{,}096}t^8 \\ & + \frac{14{,}051}{589{,}824}t^9 - \frac{13{,}103}{655{,}360}t^{10} + \frac{98{,}601}{5{,}767{,}168}t^{11} - \frac{15{,}565}{1{,}048{,}576}t^{12} \\ & + \frac{1{,}423{,}159}{109{,}051{,}904}t^{13} - \frac{1{,}361{,}617}{117{,}440{,}512}t^{14} + \frac{10{,}461{,}043}{1{,}006{,}632{,}960}t^{15} - \frac{1{,}259{,}743}{134{,}217{,}728}t^{16} \\ & + \frac{623{,}034{,}403}{73{,}014{,}444{,}032}t^{17} - \frac{603{,}217{,}979}{77{,}309{,}411{,}328}t^{18} + \frac{4{,}681{,}655{,}741}{652{,}835{,}028{,}992}t^{19}; \\ p_\gamma(t) = {}& \frac{1}{2}t + \frac{1}{4}t^2 + \frac{7}{48}t^3 + \frac{3}{32}t^4 + \frac{83}{1{,}280}t^5 + \frac{73}{1{,}536}t^6 + \frac{523}{14{,}336}t^7 + \frac{119}{4{,}096}t^8 \\ & + \frac{14{,}051}{589{,}824}t^9 + \frac{13{,}103}{655{,}360}t^{10} + \frac{98{,}601}{5{,}767{,}168}t^{11} + \frac{15{,}565}{1{,}048{,}576}t^{12} \\ & + \frac{1{,}423{,}159}{109{,}051{,}904}t^{13} + \frac{1{,}361{,}617}{117{,}440{,}512}t^{14} + \frac{10{,}461{,}043}{1{,}006{,}632{,}960}t^{15} + \frac{1{,}259{,}743}{134{,}217{,}728}t^{16} \\ & + \frac{623{,}034{,}403}{73{,}014{,}444{,}032}t^{17} + \frac{603{,}217{,}979}{77{,}309{,}411{,}328}t^{18} + \frac{4{,}681{,}655{,}741}{652{,}835{,}028{,}992}t^{19}. \end{aligned}$$

Table 1 The coefficients $a_0, a_1, \dots, a_{20}$

4,770,923,133,534,208	45,577,737,845,735,424	208,413,045,121,875,968
606,289,729,726,119,936	1,257,880,992,601,669,632	1,977,530,020,741,201,920
2,443,128,584,682,799,104	2,427,671,832,468,406,272	1,969,549,944,783,110,144
1,316,806,964,788,476,928	729,144,476,396,464,128	334,812,607,979,053,568
127,214,924,071,312,896	39,763,039,787,401,392	10,120,564,836,367,104
2,064,597,817,622,592	329,565,434,362,848	39,662,337,834,126
3,384,693,012,652	182,584,573,899	4,681,655,741

Table 2 The coefficients $b_0, b_1, \dots, b_{19}$

816,042,556,423	−8,304,263,424,095	57,076,977,341,817
−261,518,771,366,961	849,290,412,683,676	−2,026,905,939,920,124
3,601,042,197,273,780	−4,689,184,725,197,076	4,130,366,508,898,578
−1,578,461,017,820,258	−1,834,983,544,254,146	4,260,109,227,548,850
−4,661,988,848,655,060	3,484,106,242,692,852	−1,908,076,678,782,012
773,661,301,050,972	−227,689,188,034,785	46,209,023,561,673
−5,802,339,420,719	340,462,481,719	

We shall use a slightly modified polynomial $\tilde{p}_\gamma$ given by

$$\tilde{p}_\gamma(t) = p_\gamma(t) + (1/6)t^{19}.$$

Note that a term was added to p_γ, because it can be seen that $\gamma(t) > p_\gamma(t)$ for $t > 0$ sufficiently small. The coefficient 1/6 was found using some estimations which are omitted because they are not essential for the proof.

We shall prove that:

(i) $\beta(t) < p_\beta(t)$, $0 < t < 1$;

(ii) $\gamma(t) < \tilde{p}_\gamma(t)$, $0 < t < r$;

(iii) $4p_\beta(t)\tilde{p}_\gamma(t) < \alpha(t)$, $0 < t < 1$.

We denote by $f_1(t) = p_\beta(t) - \beta(t)$, $f_2(t) = f_1'(t) = p_\beta'(t) - \frac{1}{(2+t)\sqrt{1+t}}$. We substitute $t = (1+s)^2 - 1$, $0 < s < \sqrt{2} - 1$, and we get $f_2((1+s)^2 - 1) = \frac{s^{19}}{34{,}359{,}738{,}368(2+2s+s^2)(1+s)} F_2(s)$, where $F_2(s) = \sum_{n=0}^{20} a_n s^n$ is a polynomial of 20th degree whose coefficients are given in Table 1, a_0, a_1, a_2 etc. in rows.

It follows that $f_2(t) > 0$, because $F_2(s)$ has positive coefficients. Since $f_1(0) = 0$, we have $f_1(t) > 0$ on $(0,1)$ and (i) is proved.

We proceed similarly for $g_1(t) = \tilde{p}_\gamma(t) - \gamma(t)$, $g_2(t) = g_1'(t) = \tilde{p}_\gamma'(t) - \frac{1}{(t-2)\sqrt{1-t}}$.

We substitute $t = 1 - s^2$, $0 < s < 1$, and get $g_2(1 - s^2) = \frac{1}{103{,}079{,}215{,}104(1+s^2)s} G_2(s)$, where $G_2(s) = -103{,}079{,}215{,}104 + \sum_{n=0}^{19} b_n s^{2n+1}$ is a polynomial of 39th degree, the coefficients b_n being given in Table 2, three in a row.

By using the Sturm sequence for the polynomial $G_2(s)$ as stated at the end of the Introduction, both functions (in *Maple* and in *Maxima*) return two roots in $(0,1]$. Since $G_2(1) = 1$, we find that $G_2(s)$ has a unique root in $(0,1)$. It follows that $g_2(t)$ has also a unique root $t_1 \in (0,1)$, hence $g_2(t) > 0$ on $(0,t_1)$, and $g_2(t) < 0$ on $(t_1,1)$. Therefore $g_1(t) > \min(g_1(0), g_1(r))$ on $(0,r)$. But $g_1(0) = 0$ and

$$g_1(r) = \frac{7{,}545{,}035{,}064{,}001{,}924{,}896{,}095{,}274{,}314{,}681{,}659{,}544{,}598{,}682{,}938{,}990{,}892{,}623{,}399{,}131}{5{,}242{,}022{,}432{,}353{,}119{,}161{,}548{,}800{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000{,}000} - \arcsin\left(\frac{199}{201}\right) = 0.00972\ldots > 0,$$

hence $g_1(t) > \min(g_1(0), g_1(r)) = 0$ and (ii) is proved.

We consider now the function $h_1(t) = \alpha(t) - 4p_\beta(t)\tilde{p}_\gamma(t)$ and its series expansion $h_1(t) = t^6 h_2(t) + O(t^{40})$, where

$$\begin{aligned} h_2(t) = {} & 1/48 + \frac{11}{480}t^2 + \frac{547}{26{,}880}t^4 + \frac{199}{11{,}520}t^6 + \frac{207{,}401}{14{,}192{,}640}t^8 + \frac{920{,}021}{73{,}801{,}728}t^{10} \\ & + \frac{95{,}240{,}443}{8{,}856{,}207{,}360}t^{12} - \frac{8{,}128{,}804{,}483}{25{,}092{,}587{,}520}t^{14} + 1/6t^{15} \\ & - \frac{155{,}266{,}321{,}334{,}306{,}053}{1{,}999{,}672{,}863{,}904{,}235{,}520}t^{16} \\ & + 1/16t^{17} - \frac{2{,}624{,}741{,}122{,}147{,}393{,}885}{110{,}381{,}942{,}087{,}513{,}800{,}704}t^{18} + \frac{73}{2{,}304}t^{19} \\ & - \frac{374{,}493{,}942{,}658{,}190{,}094{,}451}{58{,}870{,}369{,}113{,}340{,}693{,}708{,}800}t^{20} \\ & + \frac{119}{6{,}144}t^{21} + \frac{55{,}315{,}568{,}609{,}924{,}639{,}053}{117{,}740{,}738{,}226{,}681{,}387{,}417{,}600}t^{22} + \frac{13{,}103}{983{,}040}t^{23} \\ & + \frac{34{,}201{,}104{,}415{,}289{,}943{,}432{,}424{,}777}{9{,}833{,}706{,}456{,}692{,}429{,}477{,}117{,}952{,}000}t^{24} + \frac{15{,}565}{1{,}572{,}864}t^{25} \\ & + \frac{38{,}113{,}578{,}427{,}551{,}231{,}317{,}881}{7{,}820{,}295{,}659{,}001{,}835{,}851{,}612{,}160}t^{26} \\ & + \frac{1{,}361{,}617}{176{,}160{,}768}t^{27} + \frac{50{,}378{,}210{,}487{,}327{,}721{,}746{,}099{,}089}{9{,}147{,}580{,}300{,}695{,}808{,}976{,}458{,}088{,}448}t^{28} \\ & + \frac{1{,}259{,}743}{201{,}326{,}592}t^{29} + \frac{82{,}855{,}982{,}945{,}562{,}549{,}731{,}871{,}465{,}739}{14{,}407{,}438{,}973{,}595{,}899{,}137{,}921{,}489{,}305{,}600}t^{30} \\ & + \frac{603{,}217{,}979}{115{,}964{,}116{,}992}t^{31} + \frac{47{,}775{,}326{,}983{,}451{,}755{,}017{,}721{,}677{,}497}{8{,}258{,}730{,}450{,}127{,}109{,}454{,}998{,}326{,}476{,}800}t^{32}. \end{aligned}$$

Using the Sturm sequence as before we find that the polynomial h_2 has no roots in $(0,1)$ and $t^6 h_2(t) > 0$.

Now we write $h_1(t) = t^6 h_2(t) + \sum_{n=40}^{\infty} q_n t^n$. We notice that all the coefficients q_n, $n \geq 40$ are in fact the coefficients of the series expansion of $\alpha(t) = (\arcsin t)^2$, due to the fact that $p_\beta \tilde{p}_\gamma$ is polynomial of degree less that 40. It is known that these are all positive (those of odd order being null), namely

$$q_{2k} = \frac{2^{2k-2}((k-1)!)^2}{(2k-1)!k}$$

(see [11], p.61).

It follows that $h_1(t) > 0$, hence (iii) holds.

From (i)-(iii) it follows that (5) holds also on $(0,r)$ and the proof is complete. □

Remark 8 We have chosen the form of the polynomials p_β, p_γ and $\tilde{p}_\gamma$ dealing with three parameters: the degree of the polynomials m ($m = 19$), $r \in (0,1)$ ($r = 0.995$) and the coefficient c of the supplementary term in $\tilde{p}_\gamma$ ($c = 1/6$). The value for r has been determined so that (6) is true, and the degree of the polynomials not too high (which would happened if we let $r = 1$). The parameter r being fixed, c was related only to m, in such a way that (ii) to be true. Finally, the choice of the three parameters must make (iii) to be fulfilled.

The tests we performed showed that the degree of the polynomials cannot be less than 19, maybe this can be achieved by modifying slightly r, but it seems that the degree cannot be much smaller.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

Both authors jointly worked, read and approved the final manuscript.

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