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On weak exponential expansiveness of skew-evolution semiflows in Banach spaces

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Abstract
The aim of this paper is to give several characterizations for weak exponential expansiveness properties of skew-evolution semiflows in Banach spaces. Variants for weak exponential expansiveness of some well-known results in uniform exponential stability theory (Datko (1973)) and exponential instability theory (Lupa (2010), Megan *et al.* (2008)) are obtained.

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1 Introduction

It is well known that in recent years, the exponential stability theory of one parameter semigroups of operators and evolution operators has witnessed significant development. A number of long-standing open problems have been solved and the theory seems to have obtained a certain degree of maturity. One of the most important results of the stability theory is due to Datko, who proved in 1970 in [1] that a strongly continuous semigroup of operators $\{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if for each vector x from the Banach space X, the function $t\to\|T(t)x\|$ lies in $L^2(\mathbb{R}_+)$. Later, Pazy generalizes the result in [2] for $L^p(\mathbb{R}_+)$, $p\geq 1$. In 1973, Datko [3] generalized the results above, and proved that an evolutionary process $\mathcal{U}=\{U(t,s)\}_{t\geq s\geq 0}$ with uniform exponential growth is uniformly exponentially stable if and only if there exists an exponent $p\geq 1$ such that $\sup_{s\geq 0}\int_s^\infty\|U(t,s)x\|^p\,dt<\infty$, for each $x\in X$. This result was improved by Rolewicz in 1986 (see [4]). In [5] and [6], the authors generalized the results above in the case of C_0-semigroups and evolutionary process, respectively, and presented a unified treatment in terms of Banach function spaces. In [7], the property of nonuniform exponential stability has been studied by L. Barreira and C. Valls. In addition, the weak exponential stability of evolution operators in Banach spaces has been investigated and several important results have been obtained by Lupa, Megan and Popa in [8].

Since the existence problem of exponential expansiveness of evolution equations is distinct compared to the studies devoted to stability and to dichotomy, respectively, exponential expansiveness is a powerful tool when people analyze the asymptotic behavior of dynamical systems. In the last few years, new concepts of exponential expansiveness and in particular, of exponential instability, have been introduced and characterized (see [9–20]). For instance, in [16] Megan and his partners obtained some necessary and sufficient conditions for uniform exponential instability of linear skew-product semiflows in terms of Banach sequence spaces and Banach function spaces. In [12] and [15], the cases of uniform exponential instability has been considered for evolution families and linear skew-product flows, respectively. Additionally, in [19] Lupa considered a weaker notion of instability for evolution operators, thus some necessary and sufficient characterizations for weak exponential instability of evolution operators were obtained.

The concept of skew-evolution semiflows, introduced and characterized by Stoica and Megan in [17] by means of evolution semiflows and cocycles, seems to be more appropriate for the study of asymptotic behaviors of evolution equations. They depend on three variables, contrary to a skew-product semiflow or an evolution operator, for which they are generalizations and which depend only on two. The exponential instability and uniform exponential stability for skew-evolution semiflows are studied by Stoica and Megan in [17] and [21], respectively.

In the present paper, we introduce the concept of weak exponential expansiveness for skew-evolution semiflows which is an extension of classical concept of exponential expansiveness. Our main objective is to give some characterizations for weak exponential expansiveness properties of skew-evolution semiflows in Banach spaces, and variants for weak exponential expansiveness of some well-known results in uniform exponential stability theory (Datko [3]) and exponential instability theory (Lupa [19], Megan and Stoica [17]) are obtained. As applications we obtain characterizations of the concepts in terms of Lyapnov functions. We note that in our proof we don't need to assume the strong continuity of skew-evolution semiflows.

2 Preliminaries

Let (X,d) be a metric space, V a Banach spaces. The norm on V and on the space $\mathcal{B}(V)$ of all bounded operators on V will be denoted by $\|\cdot\|$. We denote $T=\{(t,t_0)\in\mathbb{R}_+^2 : t\geq t_0\geq 0\}$ and $Y=X\times V$. I is the identity operator.

Definition 2.1 (see [21]) A mapping $\varphi : T\times X\to X$ is called evolution semiflow on X if following properties are satisfied:

(es1) $\varphi(t,t,x)=x$, $\forall(t,x)\in\mathbb{R}_+\times X$;

(es2) $\varphi(t,r,\varphi(r,t_0,x))=\varphi(t,t_0,x)$, $\forall(t,r),(r,t_0)\in T$, $\forall x\in X$.

Definition 2.2 (see [21]) A mapping $\Phi : T\times X\to\mathcal{B}(V)$ is called evolution cocycle over an evolution semiflow φ if it satisfies following properties:

(ec1) $\Phi(t,t,x)=I$, $\forall t\geq 0$, $\forall x\in X$;

(ec2) $\Phi(t,r,\varphi(r,t_0,x))\Phi(r,t_0,x)=\Phi(t,t_0,x)$,$\forall(t,r),(r,t_0)\in T$, $\forall x\in X$.

Definition 2.3 (see [21]) A mapping $C : T\times Y\to Y$ defined by

$$C(t,r,x,v)=\big(\varphi(t,r,x),\Phi(t,r,x)v\big), \tag{1}$$

where Φ is an evolution cocycle over an evolution semiflow φ, is called a skew-evolution semiflow on Y.

Example 2.4 Let us denote $\mathcal{C}=\mathcal{C}(\mathbb{R}_+,\mathbb{R})$ the set of all continuous functions $x:\mathbb{R}_+\to\mathbb{R}$, endowed with the topology of uniform convergence on bounded sets. The set $\mathcal{C}$ is metrizable with respect to the metric

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x,y)}{1 + d_n(x,y)}, \quad \text{where } d_n(x,y) = \sup_{t\in[-n,n]} \bigl|x(t) - y(t)\bigr|.$$

A function $f : \mathbb{R}_+ \to [1,\infty)$ defined by $f(t) = e^{-t} + 1$, $t \geq 0$. We denote $f_t(\tau) = f(t + \tau)$, $\forall t, \tau \in \mathbb{R}_+$. Let X be the closure of the set $\{f_t, t \in \mathbb{R}_+\}$ in $\mathcal{C}$. Then (X,d) is a metric space and the mapping

$$\varphi : T \times X \to X, \quad \varphi(t,r,x) = x_{t-r}$$

is an evolution semiflow on X. Let $V = \mathbb{R}$. We consider $\Phi : T \times X \to \mathcal{B}(V)$ given by

$$\Phi(t,r,x)v = e^{\int_r^t x(\tau-r)\,d\tau} v,$$

which is an evolution cocycle and $C = (\varphi, \Phi)$ is a skew-evolution semiflow on Y.

Remark 2.5 The skew-evolution semiflows are generalizations of the evolution operators and of the skew-product semiflows (*cf.* [21], Example 2 and 3).

Definition 2.6 A skew-evolution semiflow $C = (\varphi, \Phi)$ is called with exponential decay if there are $M, \omega > 0$ such that

$$\bigl\|\Phi(t,t_0,x_0)v_0\bigr\| \geq Me^{-\omega(t-r)} \bigl\|\Phi(r,t_0,x_0)v_0\bigr\|, \tag{2}$$

for all $(t,r), (r,t_0) \in T$ and all $(x_0, v_0) \in Y$.

Definition 2.7 A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be uniformly expansive if there exists a constant $N > 0$ such that

$$\bigl\|\Phi(t,t_0,x_0)v_0\bigr\| \geq N\bigl\|\Phi(r,t_0,x_0)v_0\bigr\|, \tag{3}$$

for all $(t,r), (r,t_0) \in T$ and all $(x_0, v_0) \in Y$.

Definition 2.8 A skew-evolution semiflow $C = (\varphi, \Phi)$ is said to be uniformly exponentially expansive if there are $N, \alpha > 0$ such that

$$\bigl\|\Phi(t,t_0,x_0)v_0\bigr\| \geq Ne^{\alpha(t-r)} \bigl\|\Phi(r,t_0,x_0)v_0\bigr\|,$$

for all $(t,r), (r,t_0) \in T$ and all $(x_0, v_0) \in Y$.

Remark 2.9 It is obvious that a skew-evolution semiflow $C = (\varphi, \Phi)$ is uniformly exponentially expansive if and only if there are $N, \alpha > 0$ such that

$$\bigl\|\Phi(t,t_0,x_0)v_0\bigr\| \geq Ne^{\alpha(t-t_0)} \|v_0\|, \tag{4}$$

for all $(t, t_0, x_0, v_0) \in T \times Y$.

Remark 2.10 If a skew-evolution semiflow is uniformly exponentially expansive then it is uniformly expansive. The converse is not necessarily valid. To show this we consider the following example.

Example 2.11 We consider $X = \mathbb{R}_+$, $V = \mathbb{R}$ and a non-decreasing and bounded function $f : \mathbb{R}_+ \to [1, \infty)$. It is obvious that the mapping $\varphi : T \times \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\varphi(t, r, x_0) = t - r + x_0, \quad (t, r, x_0) \in T \times \mathbb{R}_+$$

is an evolution semiflow on $\mathbb{R}_+$ and the mapping $\Phi : T \times \mathbb{R}_+ \to \mathcal{B}(\mathbb{R})$ given by

$$\Phi(t, r, x_0) = \frac{f(t - r + x_0)}{f(x_0)}, \quad (t, r, x_0) \in T \times \mathbb{R}_+$$

is an evolution cocycle on $\mathbb{R}$. Then the skew-evolution semiflow $C = (\varphi, \Phi)$ is uniformly expansive with $N = 1$.

On the other hand, if we assume that $C = (\varphi, \Phi)$ is uniformly exponentially expansive then there are constants $N, \alpha > 0$ such that

$$\frac{f(t - r + x_0)}{f(x_0)} \geq N e^{\alpha(t-r)}, \quad \forall (t, r, x_0) \in T \times \mathbb{R}_+.$$

From this, for $r = 0$ we obtain $f(t + x_0) \geq N e^{\alpha t} f(x_0)$ which for $t \to \infty$ gives a contradiction and hence C is not uniformly exponentially expansive.

Definition 2.12 A skew-evolution semiflow $C = (\varphi, \Phi)$ is called weakly exponentially expansive if there are $N, \alpha > 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \geq 0$ with

$$\left\| \Phi(t, t_0, x_0) v_0 \right\| \geq N e^{\alpha(t-r)} \left\| \Phi(r, t_0, x_0) v_0 \right\|, \tag{5}$$

for all $t \geq r \geq t_0$.

Remark 2.13 If a skew-evolution semiflow is uniformly exponentially expansive then it is weakly exponentially expansive.

The following example shows that the converse is not valid.

Example 2.14 We consider the metric space X and an evolution semiflow φ on X defined as in Example 2.11. Let $V = \mathbb{R}^2$ with the Euclidean norm, and the evolution operator $U(t, t_0) = P(t, t_0) Q(t_0)$ (also see [19], Example 11), where

$$P(t, t_0) = \begin{pmatrix} e^{t-t_0} \sin t & e^{-(t-t_0)} \cos t \\ -e^{t-t_0} \cos t & e^{-(t-t_0)} \sin t \end{pmatrix} \quad \text{and} \quad Q(t_0) = \begin{pmatrix} \cos t_0 & \sin t_0 \\ \sin t_0 & -\cos t_0 \end{pmatrix}.$$

Then the mapping $\Phi_U : T \times X \to \mathcal{B}(\mathbb{R}^2)$ given by $\Phi_U(t, t_0, x_0) = U(t, t_0)$ is an evolution cocycle on $\mathbb{R}^2$ over the evolution semiflow φ.

For every $v_0 \in \mathbb{R}^2$ there exist $\rho \geq 0$ and $t_0 \in [0, 2\pi)$ such that $v_0 = (\rho \cos t_0, \rho \sin t_0)^T$. It is easy to see that

$$\Phi_U(t, t_0, x_0) v_0 = P(t, t_0) Q(t_0) v_0 = P(t, t_0)(\rho, 0)^T = \left(\rho e^{t-t_0} \sin t, -\rho e^{t-t_0} \cos t\right)^T$$

and hence

$$\|\Phi_U(t,t_0,x_0)v_0\| = \rho e^{t-t_0} = e^{t-r}\|\Phi_U(r,t_0,x_0)v_0\|,$$

for all $t \geq r \geq t_0$, which proves that the skew-evolution semiflow $C = (\varphi, \Phi)$ is weakly exponentially expansive.

On the other hand, we observe that for $y_0 = (\sin t_0, -\cos t_0)^T$,

$$\Phi_U(t,t_0,x_0)y_0 = P(t,t_0)(0,1)^T = \left(e^{-(t-t_0)}\cos t, e^{-(t-t_0)}\sin t\right)^T$$

and hence

$$\|\Phi_U(t,t_0,x_0)y_0\| = e^{-(t-t_0)} = e^{-(t-r)}\|\Phi_U(r,t_0,x_0)y_0\|,$$

which shows that C is not uniformly exponentially expansive.

Definition 2.15 A skew-evolution semiflow $C = (\varphi, \Phi)$ is called weakly exponentially expansive in the Barreira-Valls sense if there are $N, \alpha > 0$ and $\beta \geq 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \geq 0$ with

$$e^{\beta r}\|\Phi(t,t_0,x_0)v_0\| \geq N e^{\alpha(t-r)}\|\Phi(r,t_0,x_0)v_0\|, \tag{6}$$

for all $t \geq r \geq t_0$.

Definition 2.16 (see [17]) A skew-evolution semiflow $C = (\varphi, \Phi)$ is called strongly measurable if the mapping $t \mapsto \|\Phi(t,t_0,x_0)v_0\|$ is measurable on $[t_0, \infty)$ for all $(t_0, x_0, v_0) \in \mathbb{R}_+ \times Y$.

Definition 2.17 A mapping $\mathcal{L} : T \times Y \to \mathbb{R}$ is said to be a Lyapunov function for the skew-evolution semiflow $C = (\varphi, \Phi)$ if there is a constant $a \geq 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \geq 0$ with

$$\mathcal{L}(t,t_0,x_0,v_0) + \int_r^t e^{a(t-s)}\|\Phi(s,t_0,x_0)v_0\|^2\,ds \leq \mathcal{L}(r,t_0,x_0,v_0), \tag{7}$$

for all $t \geq r \geq t_0$.

3 The main results

Proposition 3.1 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is weakly exponentially expansive if and only if there exists a decreasing function $f : [0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} f(t) = 0$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ with the property*

$$\|\Phi(r,t_0,x_0)v_0\| \leq f(t-r)\|\Phi(t,t_0,x_0)v_0\|, \tag{8}$$

for all $t \geq r \geq t_0$.

Proof Necessity. It is a simple verification for $f(t) = \frac{1}{N}e^{-\alpha t}$, where N and α are given by Definition 2.12.

Sufficiency. According to the property of function f, there exists a constant $\delta > 0$ such that $f(\delta) < 1$. From the hypothesis we find that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ satisfying relation (8). For every $t \geq r \geq t_0$ there are $n \in \mathbb{N}$ and $l \in [0, \delta)$ such that $t - r = n\delta + l$. Then the following inequalities:

$$\begin{aligned}
\|\Phi(r, t_0, x_0)v_0\| &\leq f(l)\|\Phi(t - n\delta, t_0, x_0)v_0\| \\
&\leq f(l)f(\delta)\|\Phi(t - (n-1)\delta, t_0, x_0)v_0\| \\
&\leq \cdots \\
&\leq f(l)[f(\delta)]^n\|\Phi(t, t_0, x_0)v_0\| \\
&\leq \frac{f(0)}{f(\delta)}[f(\delta)]^{n+1}\|\Phi(t, t_0, x_0)v_0\| \\
&\leq \frac{1}{N}e^{-\alpha(t-r)}\|\Phi(t, t_0, x_0)v_0\|
\end{aligned}$$

hold for all $t \geq r \geq t_0$, where we have denoted $N = \frac{f(\delta)}{f(0)}$ and $\alpha = -\frac{\ln f(\delta)}{\delta}$.

Finally, it follows that $C = (\varphi, \Phi)$ is weakly exponentially expansive. □

Corollary 3.2 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is weakly exponentially expansive if and only if there exists a non-decreasing function $g : [0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} g(t) = +\infty$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ with the property*

$$g(t - r)\|\Phi(r, t_0, x_0)v_0\| \leq \|\Phi(t, t_0, x_0)v_0\|, \tag{9}$$

for all $t \geq r \geq t_0$.

Theorem 3.3 *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential decay. Then C is weakly exponentially expansive if and only if there are $p > 0$ and $L > 0$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ with*

$$\int_{t_0}^{t} \|\Phi(s, t_0, x_0)v_0\|^p\, ds \leq L\|\Phi(t, t_0, x_0)v_0\|^p, \tag{10}$$

for all $t \geq t_0$.

Proof Necessity. If C is weakly exponentially expansive then from Definition 2.12 it follows that there are $N, \alpha > 0$ with the property that for all $(x_0, v_0) \in Y$ there exists $t_0 \geq 0$ such that

$$\begin{aligned}
&\int_{t_0}^{t} \|\Phi(s, t_0, x_0)v_0\|^p\, ds \\
&\quad\leq N^{-p}\int_{t_0}^{t} e^{-\alpha p(t-s)}\, ds\|\Phi(t, t_0, x_0)v_0\|^p \\
&\quad\leq L\|\Phi(t, t_0, x_0)v_0\|^p,
\end{aligned}$$

for all $t \geq t_0$, where $p > 0$ is fixed and $L = \frac{N^{-p}}{\alpha p}$.

Sufficiency. We assume that there are $p > 0$ and $L > 0$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ satisfying inequality (10). Let $t \geq r \geq t_0$. If $t \geq r + 1$ we have

$$\begin{aligned} L\left\|\Phi(t,t_0,x_0)v_0\right\|^p &\geq \int_{t_0}^{t}\left\|\Phi(s,t_0,x_0)v_0\right\|^p ds \geq \int_{r}^{t}\left\|\Phi(s,t_0,x_0)v_0\right\|^p ds \\ &\geq M^p\int_r^t e^{-\omega p(s-r)}\,ds\left\|\Phi(r,t_0,x_0)v_0\right\|^p \\ &= M^p\int_0^{t-r} e^{-\omega p\tau}\,d\tau\left\|\Phi(r,t_0,x_0)v_0\right\|^p \\ &\geq M^p\int_0^{1} e^{-\omega p\tau}\,d\tau\left\|\Phi(r,t_0,x_0)v_0\right\|^p \\ &= \frac{1-e^{-p\omega}}{p\omega}M^p\left\|\Phi(r,t_0,x_0)v_0\right\|^p \end{aligned}$$

and for $t \in [r, r+1)$ we have

$$\left\|\Phi(t,t_0,x_0)v_0\right\|^p \geq e^{-p\omega}M^p\left\|\Phi(r,t_0,x_0)v_0\right\|^p,$$

where $M, \omega > 0$ are given by Definition 2.6.

Hence

$$\left\|\Phi(t,t_0,x_0)v_0\right\|^p \geq K\left\|\Phi(r,t_0,x_0)v_0\right\|^p, \tag{11}$$

for all $t \geq r \geq t_0$, where $K = M^p[e^{-p\omega} + (1 - e^{-p\omega})/p\omega L]$.

On the other hand

$$\begin{aligned} L\left\|\Phi(t,t_0,x_0)v_0\right\|^p &\geq \int_{t_0}^{t}\left\|\Phi(s,t_0,x_0)v_0\right\|^p ds \geq \int_{r}^{t}\left\|\Phi(s,t_0,x_0)v_0\right\|^p ds \\ &\geq K(t-r)\left\|\Phi(r,t_0,x_0)v_0\right\|^p, \end{aligned} \tag{12}$$

for all $t \geq r \geq t_0$.

Adding up (11) and (12) we obtain

$$\left(1 + L^{1/p}\right)\left\|\Phi(t,t_0,x_0)v_0\right\| \geq K^{1/p}\left[1 + (t-r)^{1/p}\right]\left\|\Phi(r,t_0,x_0)v_0\right\|,$$

for all $t \geq r \geq t_0$. According to Corollary 3.2, C is weakly exponentially expansive, which ends the proof. □

Theorem 3.4 *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential decay. Then C is weakly exponentially expansive if and only if there are $\mathcal{L} : T \times Y \to \mathbb{R}_-$ a Lyapunov function for C and a constant $b > 0$ such that*

$$\left|\mathcal{L}(t,t_0,x_0,v_0)\right| \leq b\left\|\Phi(t,t_0,x_0)v_0\right\|^2, \tag{13}$$

for all $t \geq t_0$.

Proof Necessity. Let $a = 0$. We consider the application $\mathcal{L} : T \times Y \to \mathbb{R}_-$,

$$\mathcal{L}(t,r,x,v) = -\int_r^t e^{a(t-s)} \big\| \Phi(s,r,x)v \big\|^2 \, ds \le 0.$$

Then from Definition 2.12 we find that there are $N, \alpha > 0$, and for every $(x_0, v_0) \in Y$ there is $t_0 \ge 0$ with

$$\big|\mathcal{L}(t,t_0,x_0,v_0)\big| = \int_{t_0}^t \big\| \Phi(s,t_0,x_0)v_0 \big\|^2 \, ds \le b \big\| \Phi(t,t_0,x_0)v_0 \big\|^2,$$

for all $t \ge t_0$, where $b = \frac{1}{2\alpha N^2}$. It is easy to see that

$$\mathcal{L}(t,t_0,x_0,v_0) + \int_r^t e^{a(t-s)} \big\| \Phi(s,t_0,x_0)v_0 \big\|^2 \, ds - \mathcal{L}(r,t_0,x_0,v_0) \le 0,$$

for all $t \ge r \ge t_0$. Hence $\mathcal{L}$ is a Lyapunov function for C such that the relation (13) is true.

Sufficiency. We assume that there are $\mathcal{L} : T \times Y \to \mathbb{R}_-$ a Lyapunov function for C and a constant $b > 0$ such that the relation (13) hold.

Then

$$\begin{aligned}
\int_{t_0}^t \big\| \Phi(s,t_0,x_0)v_0 \big\|^2 \, ds &\le \int_{t_0}^t e^{a(t-s)} \big\| \Phi(s,t_0,x_0)v_0 \big\|^2 \, ds \\
&\le \mathcal{L}(t_0,t_0,x_0,v_0) - \mathcal{L}(t,t_0,x_0,v_0) \\
&\le -\mathcal{L}(t,t_0,x_0,v_0) = \big|\mathcal{L}(t,t_0,x_0,v_0)\big| \\
&\le b \big\| \Phi(t,t_0,x_0)v_0 \big\|^2,
\end{aligned}$$

for all $t \ge t_0$, where $a \ge 0$ is given by Definition 2.17. By Theorem 3.3 we conclude that C is weakly exponentially expansive. □

Proposition 3.5 *A skew-evolution semiflow $C = (\varphi, \Phi)$ is weakly exponentially expansive in the Barreira-Valls sense if and only if there are $N > 0$, $\lambda > \mu \ge 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \ge 0$ with*

$$e^{\mu t} \big\| \Phi(t,t_0,x_0)v_0 \big\| \ge N e^{\lambda(t-r)} \big\| \Phi(r,t_0,x_0)v_0 \big\|, \tag{14}$$

for all $t \ge r \ge t_0$.

Proof Necessity. It follows by a simple verification for $\mu = \beta$ and $\lambda = \alpha + \beta$, where constants $\alpha > 0$ and $\beta \ge 0$ are given by Definition 2.15.

Sufficiency. From the hypothesis, there are $N > 0$, $\lambda > \mu \ge 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \ge 0$ satisfying

$$\begin{aligned}
N \big\| \Phi(r,t_0,x_0)v_0 \big\| &\le e^{\mu t} e^{-\lambda(t-r)} \big\| \Phi(t,t_0,x_0)v_0 \big\| \\
&= e^{\mu r} e^{-(\lambda-\mu)(t-r)} \big\| \Phi(t,t_0,x_0)v_0 \big\|,
\end{aligned}$$

for all $t \ge r \ge t_0$, which implies that C is weakly exponentially expansive in the Barreira-Valls sense with $\alpha = \lambda - \mu$ and $\beta = \mu$. □

Theorem 3.6 *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential decay. Then C is weakly exponentially expansive in the Barreira-Valls sense if and only if there are $L, \alpha > 0$, $p > 0$ and $\beta \geq 0$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ with*

$$\int_{t_0}^{t} e^{p(\alpha+\beta)(t-s)} \left\| \Phi(s, t_0, x_0)v_0 \right\|^p ds \leq L e^{p\beta t} \left\| \Phi(t, t_0, x_0)v_0 \right\|^p, \tag{15}$$

for all $t \geq t_0$.

Proof Necessity. If C is weakly exponentially expansive in the Barreira-Valls sense, then by Proposition 3.5 there are $N > 0$, $\lambda > \mu \geq 0$ such that for all $(x_0, v_0) \in Y$ there exists $t_0 \geq 0$ with

$$\begin{aligned}
&\int_{t_0}^{t} e^{p(\alpha+\beta)(t-s)} \left\| \Phi(s, t_0, x_0)v_0 \right\|^p ds \\
&\quad \leq N^{-p} e^{p\mu t} \int_{t_0}^{t} e^{-(\lambda-\alpha-\beta)p(t-s)} ds \left\| \Phi(t, t_0, x_0)v_0 \right\|^p \\
&\quad \leq L e^{p\beta t} \left\| \Phi(t, t_0, x_0)v_0 \right\|^p,
\end{aligned}$$

for all $t \geq r \geq t_0$, where $p > 0$ is fixed, $\beta = \mu$, $\alpha \in (0, \lambda - \mu)$ and $L = \frac{N^{-p}}{p(\lambda-\alpha-\beta)}$.

Sufficiency. We assume that there are $L, \alpha > 0$, $p > 0$ and $\beta \geq 0$ such that for every $(x_0, v_0) \in Y$ there is $t_0 \geq 0$ satisfying inequality (15). Let $t \geq r \geq t_0$. If $t \geq r + 1$ then

$$\begin{aligned}
&M^p e^{-p(\alpha+\beta+\omega)} e^{p(\alpha+\beta)(t-r)} \left\| \Phi(r, t_0, x_0)v_0 \right\|^p \\
&\quad = \int_{r}^{r+1} M^p e^{-p(\alpha+\beta+\omega)} e^{p(\alpha+\beta)(t-r)} \left\| \Phi(r, t_0, x_0)v_0 \right\|^p ds \\
&\quad \leq \int_{r}^{r+1} e^{-p(\alpha+\beta+\omega)} e^{p\omega(s-r)} e^{p(\alpha+\beta)(t-s)} e^{p(\alpha+\beta)(s-r)} \left\| \Phi(s, t_0, x_0)v_0 \right\|^p ds \\
&\quad = \int_{r}^{r+1} e^{p(\alpha+\beta+\omega)[(s-r)-1]} e^{p(\alpha+\beta)(t-s)} \left\| \Phi(s, t_0, x_0)v_0 \right\|^p ds \\
&\quad \leq \int_{r}^{r+1} e^{p(\alpha+\beta)(t-s)} \left\| \Phi(s, t_0, x_0)v_0 \right\|^p ds \\
&\quad \leq L e^{p\beta t} \left\| \Phi(t, t_0, x_0)v_0 \right\|^p,
\end{aligned}$$

and therefore

$$e^{(\alpha+\beta)r} \left\| \Phi(t, t_0, x_0)v_0 \right\| \geq M L^{-1/p} e^{-(\alpha+\beta+\omega)} e^{\alpha t} \left\| \Phi(r, t_0, x_0)v_0 \right\|,$$

where $M, \omega > 0$ are given by Definition 2.6.

We consider $t \in [r, r + 1)$. Then

$$\begin{aligned}
\left\| \Phi(r, t_0, x_0)v_0 \right\| &\leq M^{-1} e^{\omega(t-r)} \left\| \Phi(t, t_0, x_0)v_0 \right\| \\
&= M^{-1} e^{(\alpha+\beta+\omega)(t-r)} e^{-(\alpha+\beta)(t-r)} \left\| \Phi(t, t_0, x_0)v_0 \right\| \\
&\leq M^{-1} e^{(\alpha+\beta+\omega)} e^{-(\alpha+\beta)(t-r)} \left\| \Phi(t, t_0, x_0)v_0 \right\|.
\end{aligned}$$

Further we obtain

$$\begin{aligned} e^{(\alpha+\beta)r}\left\|\Phi(t,t_0,x_0)v_0\right\| &\geq Me^{-(\alpha+\beta+\omega)}e^{(\alpha+\beta)t}\left\|\Phi(r,t_0,x_0)v_0\right\| \\ &\geq Me^{-(\alpha+\beta+\omega)}e^{\alpha t}\left\|\Phi(r,t_0,x_0)v_0\right\|. \end{aligned}$$

Hence, C is weakly exponentially expansive in the Barreira-Valls sense, which ends the proof. □

Remark 3.7 Theorems 3.3 and 3.6 are the versions of the classical stability theorems and exponential instability theorems due to Datko [3], Lupa [19], Megan and Stoica [17], for weak exponential expansiveness of skew-evolution semiflows.

Theorem 3.8 *Let $C = (\varphi, \Phi)$ be a strongly measurable skew-evolution semiflow with exponential decay. Then C is weakly exponentially expansive in the Barreira-Valls sense if and only if there are $\mathcal{L} : T \times Y \to \mathbb{R}_-$ a Lyapunov function for C and constants $c > 0$, $d \geq 0$ such that*

$$\left|\mathcal{L}(t,t_0,x_0,v_0)\right| \leq ce^{dt}\left\|\Phi(t,t_0,x_0)v_0\right\|^2, \tag{16}$$

for all $t \geq t_0$.

Proof According to the conclusion of Theorem 3.6, the argumentation can be obtained as well as that of Theorem 3.4. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors completed the paper together. They also read and approved the final manuscript.

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