Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

Approximating fixed points for generalized nonexpansive mapping in CAT(0) spaces

Izhar Uddin[1], Sumitra Dalal[2*] and Mohammad Imdad[1]

*Correspondence:
mathsqueen_d@yahoo.com
[2]Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

Takahashi and Kim (Math. Jpn. 48:1-9, 1998) used the Ishikawa iteration process to prove some convergence theorems for nonexpansive mappings in Banach spaces. The aim of this paper is to prove similar results in CAT(0) spaces for generalized nonexpansive mappings, which, in turn, generalize the corresponding results of Takahashi and Kim (Math. Jpn. 48:1-9, 1998), Laokul and Panyanak (Int. J. Math. Anal. 3(25-28):1305-1315, 2009), Razani and Salahifard (Bull. Iran. Math. Soc. 37(1):235-246, 2011) and some others.

MSC: 47H10; 54H25

Keywords: CAT(0) spaces; fixed point; Δ-convergence; condition (E) and Opial property

1 Introduction

A self-mapping T defined on a bounded closed and convex subset K of a Banach space X is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in K.$$

In an attempt to construct a convergent sequence of iterates with respect to a nonexpansive mapping, Mann [1] defined an iteration method as follows: (for any $x_1 \in K$)

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \quad n \in \mathbb{N},$$

where $\alpha_n \in (0, 1)$.

In 1974, with a view to approximate the fixed point of pseudo-contractive compact mappings in Hilbert spaces, Ishikawa [2] introduced a new iteration procedure as follows (for $x_1 \in K$):

$$\begin{cases} y_n = (1 - \alpha_n)x_n + \alpha_n Tx_n, \\ x_{n+1} = (1 - \beta_n)x_n + \beta_n Ty_n, \quad n \in \mathbb{N}, \end{cases}$$

where $\alpha_n, \beta_n \in (0, 1)$.

For a comparison of the preceding two iterative schemes in one-dimensional case, we refer the reader to Rhoades [3] wherein it is shown that under suitable conditions (see part (a) of Theorem 3) the rate of convergence of the Ishikawa iteration is faster than that of the Mann iteration procedure. Iterative techniques for approximating fixed points of nonexpansive single-valued mappings have been investigated by various authors using the Mann as well as Ishikawa iteration schemes. By now, there exists an extensive literature on the iterative fixed points for various classes of mappings. For an up-to-date account of the literature on this theme, we refer the readers to Berinde [4].

In 1998, Takahashi and Kim [5] consider the Ishikawa iteration procedure described as

$$\begin{cases} y_n = \alpha_n Tx_n + (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n Ty_n + (1-\beta_n)x_n \quad \text{for } n \geq 1, \end{cases} \tag{1}$$

where the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0,1]$ such that one of the following holds:

(i) $\alpha_n \in [a,b]$ and $\beta_n \in [0,b]$ for some a, b with $0 < a \leq b < 1$,

(ii) $\alpha_n \in [a,1]$ and $\beta_n \in [a,b]$ for some a, b with $0 < a \leq b < 1$.

Utilizing the forgoing iterative scheme, Takahashi and Kim [5] proved weak as well as strong convergence theorems for a nonexpansive mapping in Banach spaces.

Recently, García-Falset *et al.* [6] introduced two generalizations of nonexpansive mappings which in turn include Suzuki generalized nonexpansive mappings contained in [7] and also utilized the same to prove some fixed point theorems.

The following definitions are relevant to our subsequent discussions.

Definition 1.1 ([6]) Let C be the nonempty subset of a Banach space X and $T : C \to X$ be a single-valued mapping. Then T is said to satisfy the condition (E_μ) (for some $\mu \geq 1$) if for all $x, y \in C$

$$\|x - Ty\| \leq \mu\|x - Tx\| + \|x - y\|.$$

We say that T satisfies the condition (E) whenever T satisfies the condition (E_μ) for some $\mu \geq 1$.

Definition 1.2 ([6]) Let C be the nonempty subset of a Banach space X and $T : C \to X$ be a single-valued mapping. Then T is said to satisfy the condition (C_λ) (for some $\lambda \in (0,1)$) if for all $x, y \in C$

$$\lambda\|x - Tx\| \leq \|x - y\| \quad \Rightarrow \quad \|Tx - Ty\| \leq \|x - y\|.$$

The following theorem is essentially due to García-Falset *et al.* [6].

Theorem 1.3 ([6]) *Let C be a convex subset of a Banach space X and $T : C \to C$ be a mapping satisfying conditions (E) and (C_λ) for some $\lambda \in (0,1)$. Further, assume that either of the following holds.*

(a) *C is weakly compact and X satisfies the Opial condition.*

(b) *C is compact.*

(c) *C is weakly compact and X is $(UCED)$.*

Then there exists $z \in C$ such that $Tz = z$.

A metric space (X, d) is a CAT(0) space if it is geodesically connected and every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. It is well known that any complete, simply connected Riemannian manifold having nonpositive sectional curvature is a CAT(0) space. Other examples are pre-Hilbert spaces, R-trees, the Hilbert ball (see [8]) and many others. For more details on CAT(0) spaces, one can consult [9–11]. Fixed point theory in CAT(0) spaces was initiated by Kirk [12] who proved some theorems for nonexpansive mappings. Since then the fixed point theory for single-valued as well as multi-valued mappings has intensively been developed in CAT(0) spaces (*e.g.* [13–16]). Further relevant background material on CAT(0) spaces is included in the next section.

The purpose of this paper is to prove some weak and strong convergence theorems of the iterative scheme (1) in CAT(0) spaces for generalized nonexpansive mappings enabling us to enlarge the class of underlying mappings as well as the class of spaces in the corresponding results of Takahashi and Kim [5].

2 Preliminaries

In this section, to make our presentation self-contained, we collect relevant definitions and results. In a metric space (X, d), a geodesic path joining $x \in X$ and $y \in X$ is a map c from a closed interval $[0, r] \subset R$ to X such that $c(0) = x$, $c(r) = y$ and $d(c(t), c(s)) = |s - t|$ for all $s, t \in [0, r]$. In particular, the mapping c is an isometry and $d(x, y) = r$. The image of c is called a geodesic segment joining x and y, which is denoted by $[x, y]$, whenever such a segment exists uniquely. For any $x, y \in X$, we denote the point $z \in [x, y]$ by $z = (1 - \alpha)x \oplus \alpha y$, where $0 \leq \alpha \leq 1$ if $d(x, z) = \alpha d(x, y)$ and $d(z, y) = (1 - \alpha)d(x, y)$. The space (X, d) is called a geodesic space if any two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset C of X is called convex if C contains every geodesic segment joining any two points in C.

A geodesic triangle $\triangle(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points of X (as the vertices of $\triangle$) and a geodesic segment between each pair of points (as the edges of $\triangle$). A comparison triangle for $\triangle(x_1, x_2, x_3)$ in (X, d) is a triangle $\overline{\triangle}(x_1, x_2, x_3) := \triangle(\overline{x}_1, \overline{x}_2, \overline{x}_3)$ in the Euclidean plane $\mathbb{R}^2$ such that $d_{\mathbb{R}^2}(\overline{x}_i, \overline{x}_j) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$. A point $\overline{x} \in [\overline{x}_1, \overline{x}_2]$ is said to be comparison point for $x \in [x_1, x_2]$ if $d(x_1, x) = d(\overline{x}_1, \overline{x})$. Comparison points on $[\overline{x}_2, \overline{x}_3]$ and $[\overline{x}_3, \overline{x}_1]$ are defined in the same way.

A geodesic metric space X is called a CAT(0) space if all geodesic triangles satisfy the following comparison axiom (CAT(0) inequality):

Let $\triangle$ be a geodesic triangle in X and let $\overline{\triangle}$ be its comparison triangle in $\mathbb{R}^2$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $x, y \in \triangle$ and all comparison points $\overline{x}, \overline{y} \in \overline{\triangle}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\overline{x}, \overline{y}).$$

If x, y_1 and y_2 are points of CAT(0) space and y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

The above inequality is known as the (CN) inequality and was given by Bruhat and Tits [17]. A geodesic space is a CAT(0) space if and only if it satisfies the (CN) inequality. The following are some examples of CAT(0) spaces:

(i) Any convex subset of a Euclidean space $\mathbb{R}^n$, when endowed with the induced metric is a CAT(0) space.
(ii) Every pre-Hilbert space is a CAT(0) space.
(iii) If a normed real vector space X is CAT(0) space, then it is a pre-Hilbert space.
(iv) The Hilbert ball with the hyperbolic metric is a CAT(0) space.
(v) If X_1 and X_2 are CAT(0) spaces, then $X_1 \times X_2$ is also a CAT(0) space.

For detailed information regarding these spaces, one can refer to [9–11, 17].

Now, we collect some basic geometric properties which are instrumental throughout our subsequent discussions. Let X be a complete CAT(0) space and $\{x_n\}$ be a bounded sequence in X. For $x \in X$ set

$$r\bigl(x, \{x_n\}\bigr) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r\bigl(\{x_n\}\bigr) = \inf\bigl\{r(x, x_n) : x \in X\bigr\},$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as

$$A\bigl(\{x_n\}\bigr) = \bigl\{x \in X : r(x, x_n) = r\bigl(\{x_n\}\bigr)\bigr\}.$$

It is well known that in a CAT(0) space, $A(\{x_n\})$ consists of exactly one point (see Proposition 5 of [18]).

In 2008, Kirk and Panyanak [19] gave a concept of convergence in CAT(0) spaces which is the analog of the weak convergence in Banach spaces and a restriction of Lim's concepts of convergence [20] to CAT(0) spaces.

Definition 2.1 A sequence $\{x_n\}$ in X is said to Δ-converge to $x \in X$ if x is the unique asymptotic center of u_n for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim_n x_n = x$ and x is the Δ-limit of $\{x_n\}$.

Notice that given $\{x_n\} \subset X$ such that $\{x_n\}$ Δ-converges to x and, given $y \in X$ with $y \neq x$, by uniqueness of the asymptotic center we have

$$\limsup_{n\to\infty} d(x_n, x) < \limsup_{n\to\infty} d(x_n, y).$$

Thus every CAT(0) space satisfies the Opial property. Now we collect some basic facts about CAT(0) spaces which will be used throughout the text frequently.

Lemma 2.2 ([19]) *Every bounded sequence in a complete* CAT(0) *space admits a* Δ*-convergent subsequence.*

Lemma 2.3 ([21]) *If* C *is closed convex subset of a complete* CAT(0) *space and if* $\{x_n\}$ *is a bounded sequence in* C*, then the asymptotic center of* $\{x_n\}$ *is in* C.

Lemma 2.4 ([22]) *Let* (X, d) *be a* CAT(0) *space. For* $x, y \in X$ *and* $t \in [0, 1]$*, there exists a unique* $z \in [x, y]$ *such that*

$$d(x, z) = td(x, y) \quad \textit{and} \quad d(y, z) = (1 - t)d(x, y).$$

We use the notation $(1-t)x \oplus ty$ for the unique point z of the above lemma.

Lemma 2.5 *For $x, y, z \in X$ and $t \in [0,1]$ we have*

$$d\big((1-t)x \oplus ty, z\big) \le (1-t)d(x,z) + td(y,z).$$

Recently García-Falset *et al.* [6] introduced two generalizations of the condition (C) in Banach spaces. Now, we state their condition in the framework of CAT(0) spaces.

Definition 2.6 Let T be a mapping defined on a subset C of CAT(0) space X and $\mu \ge 1$, then T is said to satisfy the condition (E_μ), if (for all $x, y \in C$)

$$d(x, Ty) \le \mu d(x, Tx) + d(x,y).$$

We say that T satisfies the condition (E) whenever T satisfies the condition (E_μ) for some $\mu \ge 1$.

Definition 2.7 Let T be a mapping defined on a subset C of a CAT(0) space X and $\lambda \in (0,1)$, then T is said to satisfy the condition (C_λ) if (for all $x, y \in C$)

$$\lambda d(x, Tx) \le d(x,y) \quad \Rightarrow \quad d(Tx, Ty) \le d(x,y).$$

In the case $0 < \lambda_1 < \lambda_2 < 1$, then the condition (C_{λ_1}) implies the condition (C_{λ_2}). The following example shows that the class of mappings satisfying the conditions (E) and (C_λ) (for some $\lambda \in (0,1)$) is larger than the class of mappings satisfying the condition (C).

Example 2.8 ([6]) For a given $\lambda \in (0,1)$, define a mapping T on $[0,1]$ by

$$T(x) = \begin{cases} \frac{x}{2} & \text{if } x \neq 1, \\ \frac{1+\lambda}{2+\lambda} & \text{if } x = 1. \end{cases}$$

Then the mapping T satisfies the condition (C_λ) but it fails the condition (C_{λ_1}) whenever $0 < \lambda_1 < \lambda$. Moreover, T satisfies the condition (E_μ) for $\mu = \frac{2+\lambda}{2}$.

The following theorem is an analog of Theorem 4 in [6] to CAT(0) spaces.

Theorem 2.9 *Let C be a bounded convex subset of a complete* CAT(0) *space X. If $T : C \to C$ satisfies the condition (C_λ) on C for some $\lambda \in (0,1)$, then there exists an approximating fixed point sequence for T.*

Now, we present the Ishikawa iterative scheme in the framework of CAT(0) spaces. For $x_1 \in C$, the Ishikawa iteration is defined as

$$\begin{cases} y_n = \alpha_n Tx_n \oplus (1-\alpha_n)x_n, \\ x_{n+1} = \beta_n Ty_n \oplus (1-\beta_n)x_n \quad \text{for } n \ge 1, \end{cases} \tag{2}$$

where the sequences α_n and β_n are sequences in $[0,1]$.

The following lemma is a consequence of Lemma 2.9 of [23] which will be used to prove our main results.

Lemma 2.10 *Let X be a complete* CAT(0) *space and let $x \in X$. Suppose $\{t_n\}$ is a sequence in $[b,c]$ for some $b,c \in (0,1)$ and $\{x_n\}$, $\{y_n\}$ are sequences in X such that* $\limsup_{n\to\infty} d(x_n,x) \leq r$, $\limsup_{n\to\infty} d(y_n,x) \leq r$, *and* $\lim_{n\to\infty} d((1-t_n)x_n \oplus t_n y_n, x) = r$ *for some $r \geq 0$. Then*

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

In this paper, we prove that the sequence $\{x_n\}$ described by (2) Δ-converges to a fixed point of T if one of the following conditions holds:

$$\begin{cases} \text{(i)} \quad \alpha_n \in [a,b] \quad \text{and} \quad \beta_n \in [0,b] \quad \text{for some } a,b \text{ with } 0 < a \leq b < 1 \quad \text{or} \\ \text{(ii)} \quad \alpha_n \in [a,1] \quad \text{and} \quad \beta_n \in [a,b] \quad \text{for some } a,b \text{ with } 0 < a \leq b < 1. \end{cases} \tag{3}$$

This result is an analog of a result of the weak convergence theorem of Takahashi and Kim [5] for a generalized nonexpansive mapping in a Banach space. A strong convergence theorem is also proved. In the process, the corresponding results of Takahashi and Kim [5], Laokul and Panyanak [24], Razani and Salahifard [25], and others are generalized and improved.

3 Main results

Before proving our main results, firstly we re-write Theorem 1.3 in the setting of CAT(0) spaces which is essentially Theorem 3.2 of [13].

Theorem 3.1 *Let C be a bounded, closed, and convex subset of a complete* CAT(0) *space X. If $T: C \to C$ satisfies the conditions (E) and (C_λ) for some $\lambda \in (0,1)$, then T has a fixed point.*

Now, to accomplish our main results, we prove the following lemma.

Lemma 3.2 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and $T: C \to C$ be a mapping which satisfies the condition (C_λ) for some $\lambda \in (0,1)$. If $\{x_n\}$ is a sequence defined by* (2) *and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the condition described in* (3), *then $\lim_{n\to\infty} d(x_n,x)$ exists for all $x \in F(T)$.*

Proof Since T satisfies the condition (C_λ) (for some $\lambda \in (0,1)$) and $x \in F(T)$, we have

$$\lambda d(x,Tx) = 0 \leq d(x,y_n)$$

and

$$\lambda d(x,Tx) = 0 \leq d(x,x_n) \quad \text{for all } n \geq 1,$$

so that

$$d(Tx,Ty_n) \leq d(x,y_n) \quad \text{and} \quad d(Tx,Tx_n) \leq d(x,x_n).$$

Now consider

$$\begin{aligned} d(x_{n+1},x) &= d\big(\beta_n Ty_n \oplus (1-\beta_n)x_n, x\big) \\ &\le \beta_n d(Ty_n,x) + (1-\beta_n)d(x_n,x) \\ &\le \beta_n d(y_n,x) + (1-\beta_n)d(x_n,x) \\ &= \beta_n d\big(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, x\big) + (1-\beta_n)d(x_n,x) \\ &\le \beta_n\alpha_n d(Tx_n,x) + \beta_n(1-\alpha_n)d(x_n,x) + (1-\beta_n)d(x_n,x) \\ &\le d(x_n,x), \end{aligned}$$

which shows that the sequence $d(x_n,x)$ is decreasing and bounded below so that $\lim_{n\to\infty} d(x_n,x)$ exists. □

Lemma 3.3 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and let $T: C \to C$ satisfy the conditions* (C_λ) *and* (E) *on C. If $\{x_n\}$ is a sequence defined by* (2) *and the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the condition described in* (3), *then $F(T)$ is nonempty if and only if $\{x_n\}$ is bounded and* $\lim_{n\to\infty} d(Tx_n,x_n) = 0$.

Proof Suppose that the fixed point set $F(T)$ is nonempty and $x \in F(T)$. Then by Lemma 3.2, $\lim_{n\to\infty} d(x_n,x)$ exists and let us take it to be c besides that $\{x_n\}$ is bounded. We have

$$\lambda d(x,Tx) = 0 \le d(x,y_n)$$

and

$$\lambda d(x,Tx) = 0 \le d(x,x_n) \quad \text{for all } n \ge 1.$$

Owing to the condition (C_λ), we have $d(Tx,Ty_n) \le d(x,y_n)$ and $d(Tx,Tx_n) \le d(x,x_n)$.

Therefore, we have

$$\begin{aligned} d(Ty_n,x) &\le d(y_n,x) \\ &= d\big(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, x\big) \\ &\le \alpha_n d(Tx_n,x) + (1-\alpha_n)d(x_n,x) \\ &\le \alpha_n d(x_n,x) + (1-\alpha_n)d(x_n,x) \\ &= d(x_n,x), \end{aligned}$$

so that

$$\limsup_{n\to\infty} d(Ty_n,x) \le \limsup_{n\to\infty} d(y_n,x) \le c.$$

Also, we observe that

$$\lim_{n\to\infty} d\big(\beta_n Ty_n \oplus (1-\beta_n)x_n, x\big) = \lim_{n\to\infty} d(x_{n+1},x) = c.$$

Case I: If $0 < a \leq \beta_n \leq b < 1$ and $0 \leq \alpha_n \leq 1$, then by the foregoing discussion and by Lemma 2.10, we have $\lim_{n\to\infty} d(Ty_n, x_n) = 0$.

For each $\alpha_n \in [0, b]$,

$$\begin{aligned} d(Tx_n, x_n) &\leq d(Tx_n, y_n) + d(y_n, x_n) \\ &= d\bigl(Tx_n, \alpha_n Tx_n \oplus (1-\alpha_n)x_n\bigr) + d(y_n, x_n) \\ &\leq (1-\alpha_n)d(Tx_n, x_n) + d(y_n, x_n), \end{aligned}$$

so that

$$\alpha_n d(Tx_n, x_n) \leq d(y_n, x_n).$$

As $\alpha_n \in [0, b]$, in view of the condition (C_λ) we have $d(Tx_n, Ty_n) \leq d(x_n, y_n)$.

Now, consider

$$\begin{aligned} d(Tx_n, x_n) &\leq d(Tx_n, Ty_n) + d(Ty_n, x_n) \\ &\leq d(x_n, y_n) + d(Ty_n, x_n) \\ &\leq \alpha_n d(x_n, Tx_n) + d(Ty_n, x_n). \end{aligned}$$

Making use of the observation $(1-b)d(Tx_n, x_n) \leq (1-\alpha_n)d(x_n, Tx_n) \leq d(Ty_n, x_n)$, we have $\lim_{n\to\infty} d(Tx_n, x_n) \leq \frac{1}{1-b} \lim_{n\to\infty} d(Ty_n, x_n) = 0$.

Case 2: $0 < a \leq \beta_n \leq 1$ and $0 < a \leq \alpha_n \leq b < 1$.

Since we have $d(Tx_n, x) \leq d(x_n, x)$, for all $n \geq 1$, we get

$$\limsup_{n\to\infty} d(Tx_n, x) \leq c.$$

Consider

$$\begin{aligned} d(x_{n+1}, x) &\leq \beta_n d(Ty_n, x) + (1-\beta_n)d(x_n, x) \\ &\leq \beta_n d(y_n, x) + (1-\beta_n)d(x_n, x), \end{aligned}$$

which amounts to saying that

$$\frac{d(x_{n+1}, x) - d(x_n, x)}{\alpha_n} \leq d(y_n, x) - d(x_n, x).$$

Taking $\liminf_{n\to\infty}$ of both sides of the above inequality, we have

$$\liminf_{n\to\infty} \frac{d(x_{n+1}, x) - d(x_n, x)}{\alpha_n} \leq \liminf_{n\to\infty} \bigl(d(y_n, x) - d(x_n, x)\bigr).$$

As $\lim_{n\to\infty} d(x_{n+1}, x) = \lim_{n\to\infty} d(x_n, x) = c$, we have

$$0 \leq \liminf_{n\to\infty} \bigl(d(y_n, x) - d(x_n, x)\bigr),$$

while, owing to $d(y_n, x) - d(x_n, x) \leq 0$, we have

$$\liminf_{n\to\infty}\bigl(d(y_n, x) - d(x_n, x)\bigr) \leq 0.$$

Therefore, $\liminf_{n\to\infty}(d(y_n, x) - d(x_n, x)) = 0$. Thus, we get

$$\liminf_{n\to\infty} d(x_n, x) \leq \liminf_{n\to\infty} d(y_n, x).$$

That is, $c \leq \liminf_{n\to\infty} d(y_n, x)$. By combining the foregoing observations, we have

$$c \leq \liminf_{n\to\infty} d(y_n, x) \leq \limsup_{n\to\infty} d(y_n, x) \leq c,$$

so that

$$c = \lim_{n\to\infty} d(y_n, x) = \lim_{n\to\infty} d\bigl(\alpha_n Tx_n \oplus (1-\alpha_n)x_n, x\bigr).$$

Again in view of Lemma 2.10, we have $\lim_{n\to\infty} d(Tx_n, x_n) = 0$.

Conversely, suppose that $\{x_n\}$ is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. Let $A(\{x_n\}) = \{x\}$. Then, by Lemma 2.3, $x \in C$. As T satisfies the condition (E_μ) on C, there exists $\mu > 1$ such that

$$d(x_n, Tx) \leq \mu d(x_n, Tx_n) + d(x_n, x),$$

which implies that

$$\begin{aligned}\limsup_{n\to\infty} d(x_n, Tx) &\leq \limsup_{n\to\infty}\bigl\{\mu d(x_n, Tx_n) + d(x_n, x)\bigr\}\\ &= \limsup_{n\to\infty} d(x_n, x).\end{aligned}$$

Owing to the uniqueness of asymptotic center, $Tx = x$, so that x is fixed point of T. □

Theorem 3.4 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a mapping which satisfies conditions (C_λ) for some $\lambda \in (0,1)$ and (E) on C with $F(T) \neq \emptyset$. If the sequences $\{x_n\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ are described as in* (2) *and* (3), *then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof By Lemma 3.3, we observe that $\{x_n\}$ is a bounded sequence and

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Let $W_\omega(\{x_n\}) =: \bigcup A(\{u_n\})$, where the union is taken over all subsequence $\{u_n\}$ over $\{x_n\}$. To show the Δ-convergence of $\{x_n\}$ to a fixed point of T, we show that $W_\omega(\{x_n\}) \subset F(T)$ and $W_\omega(\{x_n\})$ is a singleton set. To show that $W_\omega(\{x_n\}) \subset F(T)$ let $y \in W_\omega(\{x_n\})$. Then there exists a subsequence $\{y_n\}$ of $\{x_n\}$ such that $A(\{y_n\}) = y$. By Lemmas 2.2 and 2.3, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that Δ-$\lim_n z_n = z$ and $z \in C$. Since $\lim_{n\to\infty} d(z_n, Tz_n) = 0$ and T satisfies the condition (E), there exists a $\mu \geq 1$ such that

$$d(z_n, Tz) \leq \mu d(z_n, Tz_n) + d(z_n, z).$$

By taking the lim sup of both sides, we have

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, Tz) &\le \limsup_{n\to\infty}\bigl\{\mu d(z_n, Tz_n) + d(z_n, z)\bigr\}\\ &\le \limsup_{n\to\infty} d(z_n, z).\end{aligned}$$

As $\Delta\text{-}\lim_n z_n = z$, by the Opial property

$$\limsup_{n\to\infty} d(z_n, z) \le \limsup_{n\to\infty} d(z_n, Tz).$$

Hence $Tz = z$, *i.e.* $z \in F(T)$. Now, we claim that $z = y$. If not, by Lemma 3.2, $\lim_n d(x_n, z)$ exists and, owing to the uniqueness of the asymptotic centers,

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, y)\\ &\le \limsup_{n\to\infty} d(y_n, y)\\ &< \limsup_{n\to\infty} d(y_n, z)\\ &= \limsup_{n\to\infty} d(x_n, z)\\ &= \limsup_{n\to\infty} d(z_n, z),\end{aligned}$$

which is a contradiction. Hence $y = z$. To assert that $W_\omega(\{x_n\})$ is a singleton let $\{y_n\}$ be a subsequence of $\{x_n\}$. In view of Lemmas 2.2 and 2.3, there exists a subsequence $\{z_n\}$ of $\{y_n\}$ such that $\Delta\text{-}\lim_n z_n = z$. Let $A(\{y_n\}) = y$ and $A(\{x_n\}) = x$. Earlier, we have shown that $y = z$. Therefore it is enough to show $z = x$. If $z \neq x$, then in view of Lemma 3.2 $\{d(x_n, z)\}$ is convergent. By uniqueness of the asymptotic centers

$$\begin{aligned}\limsup_{n\to\infty} d(z_n, z) &< \limsup_{n\to\infty} d(z_n, x)\\ &\le \limsup_{n\to\infty} d(x_n, x)\\ &< \limsup_{n\to\infty} d(x_n, z)\\ &= \limsup_{n\to\infty} d(z_n, z),\end{aligned}$$

which is a contradiction so that the conclusion follows. □

In view of Theorem 3.1, we have the following corollary of the preceding theorem.

Corollary 3.5 *Let C be a nonempty bounded, closed and convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a mapping which satisfies conditions (C_λ) for some $\lambda \in (0, 1)$ and (E) on C. If sequences $\{x_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are described by* (2) *and* (3), *then the sequence $\{x_n\}$ Δ-converges to a fixed point of T.*

Theorem 3.6 *Let C be a nonempty closed convex subset of a complete* CAT(0) *space X and $T : C \to C$ be a mapping which satisfies conditions (C_λ) for some $\lambda \in (0, 1)$ and (E) on C. Moreover, T satisfies the condition (I) with $F(T) \neq \emptyset$. If sequences $\{x_n\}$, $\{\alpha_n\}$, and $\{\beta_n\}$ are defined as in* (2) *and* (3), *respectively, then $\{x_n\}$ converges strongly to some fixed point of T.*

Proof To show that the fixed point set $F(T)$ is closed, let $\{x_n\}$ be a sequence in $F(T)$ which converges to some point $z \in C$. As

$$\lambda d(x_n, Tx_n) = 0 \le d(x_n, z),$$

in view of the condition (C_λ), we have

$$d(x_n, Tz) = d(Tx_n, Tz) \le d(x_n, z).$$

By taking the limit of both sides we obtain

$$\lim_{n\to\infty} d(x_n, Tz) \le \lim_{n\to\infty} d(x_n, z) = 0.$$

In view of the uniqueness of the limit, we have $z = Tz$, so that $F(T)$ is closed. Observe that by Lemma 3.2, we have $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. It follows from the condition (I) that

$$\lim_{n\to\infty} f\bigl(d\bigl(x_n, F(T)\bigr)\bigr) \le \lim_{n\to\infty} d(x_n, Tx_n) = 0,$$

so that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since $f : [0,\infty] \to [0,\infty)$ is a nondecreasing mapping satisfying $f(0) = 0$ for all $r \in (0,\infty)$, we have $\lim_{n\to\infty} d(x_n, F(T)) = 0$. This implies that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$d(x_{n_k}, p_k) \le \frac{1}{2^k} \quad \text{for all } k \ge 1$$

wherein $\{p_k\}$ is in $F(T)$. By Lemma 3.2, we have

$$d(x_{n_{k+1}}, p_k) \le d(x_{n_k}, p_k) \le \frac{1}{2^k},$$

so that

$$\begin{aligned} d(p_{k+1}, p_k) &\le d(p_{k+1}, x_{n_{k+1}}) + d(x_{n_{k+1}}, p_k) \\ &\le \frac{1}{2^{k+1}} + \frac{1}{2^k} < \frac{1}{2^{k-1}}, \end{aligned}$$

which implies that $\{p_k\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{p_k\}$ is a convergent sequence. Write $\lim_{k\to\infty} p_k = p$. Now, in order to show that $\{x_n\}$ converges to p let us proceed as follows:

$$d(x_{n_k}, p) \le d(x_{n_k}, p_k) + d(p_k, p) \to 0 \quad \text{as } k \to \infty,$$

so that $\lim_{k\to\infty} d(x_{n_k}, p) = 0$. Since $\lim_{n\to\infty} d(x_n, p)$ exists, we have $x_n \to p$. □

Corollary 3.7 *Let C be a nonempty bounded, closed, and convex subset of a complete* CAT(0) *space X and let $T : C \to C$ be a mapping which satisfies the conditions* (C_λ) *for some* $\lambda \in (0,1)$ *and* (E) *on C. Moreover, T satisfies the condition* (I). *If the sequences* $\{x_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are described by* (2) *and* (3), *then* $\{x_n\}$ *converges strongly to some fixed point of T.*

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The authors have contributed in this work on an equal basis. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Aligarh Muslim University, Aligarh, Uttar Pradesh 202002, India. [2]Department of Mathematics, Faculty of Science, Jazan University, Jazan, Saudi Arabia.

Acknowledgements
The authors thank the anonymous referees and handling editor of the manuscript for their valuable suggestions and fruitful comments. The first author is grateful to University Grants commission, India for the financial assistance in the form of Maulana Azad National Fellowship. The research of second author is supported by Jazan University, Saudi Arabia.

Received: 23 January 2014 Accepted: 16 April 2014 Published: 02 May 2014

References

1. Mann, WR: Mean value methods in iterations. Proc. Am. Math. Soc. **4**, 506-510 (1953)
2. Ishikawa, S: Fixed points by a new iteration method. Proc. Am. Math. Soc. **44**, 147-150 (1974)
3. Rhoades, BE: Comments on two fixed point iteration methods. J. Math. Anal. Appl. **56**(3), 741-750 (1976)
4. Berinde, V: Iterative Approximation of Fixed Points. Lecture Notes in Mathematics, vol. 1912. Springer, Berlin (2007)
5. Takahashi, W, Kim, GE: Approximating fixed points of nonexpansive mappings in Banach spaces. Math. Jpn. **48**, 1-9 (1998)
6. García-Falset, J, Llorens-Fuster, E, Suzuki, T: Fixed point theory for a class of generalized nonexpansive mappings. J. Math. Anal. Appl. **375**, 185-195 (2011)
7. Suzuki, T: Fixed point theorems and convergence theorems for some generalized nonexpansive mappings. J. Math. Anal. Appl. **340**, 1088-1095 (2008)
8. Goebel, K, Reich, S: Uniform Convexity, Hyperbolic Geometry, and Nonexpansive Mappings. Dekker, New York (1984)
9. Bridson, M, Haefliger, A: Metric Spaces of Nonpositive Curvature. Springer, Berlin (1999)
10. Brown, KS: Buildings. Springer, New York (1989)
11. Burago, D, Burago, Y, Ivanov, S: A Course in Metric Geometry. Graduate Studies in Math., vol. 33. Amer. Math. Soc., Providence (2001)
12. Kirk, WA: Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis, Malaga, Seville, 2002-2003. Colec. Abierta, vol. 64, pp. 195-225. Univ. Sevilla Secr. Publ., Seville (2003)
13. Abkar, A, Eslamian, M: Common fixed point results in CAT(0) spaces. Nonlinear Anal. **74**(5), 1835-1840 (2011)
14. Razani, A, Salahifard, H: Invariant approximation for CAT(0) spaces. Nonlinear Anal. **72**, 2421-2425 (2010)
15. Dhompongsa, S, Kaewkhao, A, Panyanak, B: On Kirk's strong convergence theorem for multivalued nonexpansive mappings on CAT(0) spaces. Nonlinear Anal. **75**(2), 459-468 (2012)
16. Nanjaras, B, Panyanaka, B, Phuengrattana, W: Fixed point theorems and convergence theorems for Suzuki-generalized nonexpansive mappings in CAT(0) spaces. Nonlinear Anal. Hybrid Syst. **4**, 25-31 (2010)
17. Bruhat, F, Tits, J: Groupes réductifs sur un corps local. I. Données radicielles valuées. Publ. Math. IHÉS **41**, 5-251 (1972)
18. Dhompongsa, S, Kirk, WA, Sims, B: Fixed points of uniformly Lipschitzian mappings. Nonlinear Anal. **65**, 762-772 (2006)
19. Kirk, WA, Panyanak, B: A concept of convergence in geodesic spaces. Nonlinear Anal. **68**, 3689-3696 (2008)
20. Lim, TC: Remarks on some fixed point theorems. Proc. Am. Math. Soc. **60**, 179-182 (1976)
21. Dhompongsa, S, Kirk, WA, Panyanak, B: Nonexpansive set-valued mappings in metric and Banach spaces. J. Nonlinear Convex Anal. **8**, 35-45 (2007)
22. Dhompongsa, S, Panyanak, B: On Δ-convergence theorems in CAT(0) space. Comput. Math. Appl. **56**, 2572-2579 (2008)
23. Laowang, W, Panyanak, B: Approximating fixed points of nonexpansive nonself mappings in CAT(0) spaces. Fixed Point Theory Appl. **2010**, Article ID 367274 (2010)
24. Laokul, T, Panyanak, B: Approximating fixed points of nonexpansive mappings in CAT(0) spaces. Int. J. Math. Anal. **3**(25-28), 1305-1315 (2009)
25. Razani, A, Salahifard, H: Approximating fixed points of generalized nonexpansive mappings. Bull. Iran. Math. Soc. **37**(1), 235-246 (2011)

10.1186/1029-242X-2014-155
Cite this article as: Uddin et al.: **Approximating fixed points for generalized nonexpansive mapping in** CAT(0) **spaces.** *Journal of Inequalities and Applications* 2014, 2014:155