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On coupled coincidence point theorems on partially ordered G -metric spaces without mixed g -monotone

Phakdi Charoensawan* and Chaiporn Thangthong

*Correspondence:
phakdi@hotmail.com
Department of Mathematics,
Faculty of Science, Chiang Mai
University, Chiang Mai, 50200,
Thailand

Abstract

In this work, we prove the existence of a coupled coincidence point theorem of nonlinear contraction mappings in G -metric spaces without the mixed g -monotone property and give some examples of a nonlinear contraction mapping, which is not applied to the existence of coupled coincidence point by using the mixed monotone property. We also show the uniqueness of a coupled coincidence point of the given mapping. Further, we apply our results to the existence and uniqueness of a coupled coincidence point of the given mapping in partially ordered G -metric spaces.

Keywords: coupled fixed point; coupled coincidence point; invariant set; mixed g -monotone; partially ordered set; G -metric

1 Introduction

The existence of a fixed point for the contraction type of mappings in partially ordered metric spaces has been studied by Ran and Reurings [1] and they established some new results for contractions in partially ordered metric spaces and presented applications to matrix equations. Later, Nieto and Rodríguez-Lopez [2, 3] and Agarwal *et al.* [4] presented some new results for contractions in partially ordered metric spaces. Examples of extensions and applications of these works see in [5–9].

The concept of coupled fixed point was introduced by Guo and Lakshmikantham [10]. Later, Bhaskar and Lakshmikantham [11] introduced the concept of mixed monotone property for contractive operators in partially ordered metric spaces. They also gave some applications in the existence and uniqueness of the coupled fixed point theorems for mappings which satisfy the mixed monotone property. Lakshmikantham and Ćirić [12] extended the results in [11] by defining the mixed g -monotone and to study the existence and uniqueness of coupled coincidence point for such mapping which satisfy the mixed monotone property in partially ordered metric space. As a continuation of this work, many authors conducted research on the coupled fixed point theory and coupled coincidence point theory in partially ordered metric spaces and different spaces. For example see [13–47].

Recently, Sintunavarat *et al.* [45, 46] proved some coupled fixed point theorems for nonlinear contractions without mixed monotone property and extended some coupled fixed point theorems of Bhaskar and Lakshmikantham [11] by using the concept of F -invariant set due to Samet and Vetro [48]. Later, in 2013, Batra and Vashistha [19] introduced the

concept of (F, g) -invariant set which is a generalization of an F -invariant set introduced by Samet and Vetro [48] and proved theorems on the existence of coupled fixed points for nonlinear contractions under c -distance in cone metric spaces having an (F, g) -invariant subset. Very recently, Charoensawan and Klanarong [25] proved theorems on the existence of coupled coincidence points in partially ordered metric spaces without mixed g -monotone property which extended some coupled fixed point theorems of Sintunavarat *et al.* [45]. They also proved uniqueness of coupled common fixed point theorems for nonlinear contractions.

In 2006, Mustafa and Sims [49] introduced the notion of a G -metric spaces as a generalization of the concept of a metric space and proved the analog of the Banach contraction mapping principle in the context of G -metric spaces. Following this initial research, many authors discussed research on the fixed point theory in partially ordered G -metric space (see, *e.g.*, [50–66]).

Recently, Jleli and Samet [54] showed the weakness of the fixed point theory in G -metric by introducing the concept of a quasi-metric space and showed that the result of Mustafa *et al.* [57] can be deduced by some well-known results in the literature in the setting of a usual (quasi) metric space. Later, Samet *et al.* [63] established some propositions to show that many fixed point theorems on (nonsymmetric) G -metric spaces given recently by many authors follow directly from well-known theorems on metric spaces. However, Karapinar and Agarwal [55] noticed that the techniques used in [54, 63] are valid if the contraction condition in the statement of the theorem can be expressed in two variables and they proved some theorems on the existence and uniqueness of a common fixed point for which the techniques of the papers [54, 63] are not applicable.

In recent times, coupled fixed point and coupled coincidence point theory has been developed in partially ordered G -metric space. Some authors have studied coupled fixed point theory. For example, Choudhury and Maity [27] proved the existence of a coupled fixed point theorem of nonlinear contraction mappings with mixed monotone property in partially ordered G -metric space. Later, Abbas *et al.* [13] extended the results of a coupled fixed point theorem for a mixed monotone mapping obtained by Choudhury and Maity [27].

On the other hand, some authors have studied coupled coincidence point theory in partially ordered G -metric space. In 2011, Aydi *et al.* [15] established coupled coincidence and coupled common fixed point results for a mixed g -monotone mapping satisfying nonlinear contractions in a partially ordered G -metric space. They generalized the results obtained by Choudhury and Maity [27]. Later, Karapinar *et al.* [34] extended the results of coupled coincidence and coupled common fixed point theorem for a mixed g -monotone mapping obtained by Aydi *et al.* [15]. As a continuation of this trend, many authors have studied coupled coincidence point and coupled common fixed point results for a mixed g -monotone mapping satisfying nonlinear contractions in a partially ordered G -metric space (see, for example, [16–18, 24, 26, 28, 34, 43, 44, 50, 66]). However, very recently, Agarwal and Karapinar [50] introduced the concept of g -ordered completeness and showed that the weaknesses of some of the coupled fixed point theorems and coupled coincidence point theorems in [13, 15, 26, 27, 43, 67] are in fact immediate consequences of well-known fixed point theorems in the literature.

In this work, we generalize and extend the coupled coincidence point theorem of non-linear contraction mappings in partially ordered G -metric spaces without the mixed g -monotone property.

2 Preliminaries

In this section, we give some definitions, proposition, examples, and remarks which are useful for the main results in this paper. Throughout this paper, (X, \leq) denotes a partially ordered set with the partial order \leq . By $x \leq y$, we mean $y \geq x$. A mapping $f : X \rightarrow X$ is said to be non-decreasing (resp., non-increasing) if, for all $x, y \in X$, $x \leq y$ implies $f(x) \leq f(y)$ (resp. $f(y) \geq f(x)$).

Definition 2.1 [49] Let X be a nonempty set, and $G : X \times X \times X \rightarrow \mathbb{R}^+$ be a function satisfying the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$.
- (G2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$.
- (G3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$.
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables).
- (G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$ (rectangle inequality).

Then the function G is called a generalized metric, or, more specially, a G -metric on X , and the pair (X, G) is called a G -metric space.

Example 2.2 Let (X, d) be a metric space. The function $G : X \times X \times X \rightarrow [0, +\infty)$, defined by $G(x, y, z) = d(x, y) + d(y, z) + d(z, x)$, for all $x, y, z \in X$, is a G -metric space on X .

Definition 2.3 [49] Let (X, G) be a G -metric space, and let (x_n) be a sequence of points of X . We say that (x_n) is G -convergent to $x \in X$ if $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$, that is, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x, x_n, x_m) < \varepsilon$, for all $n, m \geq N$. We call x the limit of the sequence and write $x_n \rightarrow x$ or $\lim_{n \rightarrow \infty} x_n = x$.

Proposition 2.4 [49] Let (X, G) be a G -metric space; the following are equivalent.

- (1) (x_n) is G -convergent to x .
- (2) $G(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow +\infty$.
- (3) $G(x_n, x, x) \rightarrow 0$ as $n \rightarrow +\infty$.
- (4) $G(x_n, x_m, x) \rightarrow 0$ as $n, m \rightarrow +\infty$.

Definition 2.5 [49] Let (X, G) be a G -metric space. A sequence (x_n) is called a G -Cauchy sequence if, for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$, for all $n, m, l \geq N$. That is, $G(x_n, x_m, x_l) \rightarrow 0$ as $n, m, l \rightarrow +\infty$.

Proposition 2.6 [49] Let (X, G) be a G -metric space, the following are equivalent:

- (1) the sequence (x_n) is G -Cauchy;
- (2) for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq N$.

Proposition 2.7 [49] Let (X, G) be a G -metric space. A mapping $f : X \rightarrow X$ is G -continuous at $x \in X$ if and only if it is G -sequentially continuous at x , that is, whenever (x_n) is G -convergent to x , $(f(x_n))$ is G -convergent to $f(x)$.

Definition 2.8 [49] A G -metric space (X, G) is called G -complete if every G -Cauchy sequence is G -convergent in (X, G) .

Definition 2.9 [27] Let (X, G) be a G -metric space. A mapping $F : X \times X \rightarrow X$ is said to be continuous if for any two G -convergent sequences (x_n) and (y_n) converging to x and y , respectively, $(F(x_n, y_n))$ is G -convergent to $F(x, y)$.

The concept of a mixed monotone property and a coupled fixed point have been introduced by Bhaskar and Lakshmikantham in [11].

Definition 2.10 [11] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$. We say F has the mixed monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad x_1 \leq x_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \leq y_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 2.11 [11] An element $(x, y) \in X \times X$ is called a coupled fixed point of a mapping $F : X \times X \rightarrow X$ if $F(x, y) = x$ and $F(y, x) = y$.

Lakshmikantham and Ćirić in [12] introduced the concept of a mixed g -monotone mapping and a coupled coincidence point.

Definition 2.12 [12] Let (X, \leq) be a partially ordered set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F has the mixed g -monotone property if for any $x, y \in X$

$$x_1, x_2 \in X, \quad g(x_1) \leq g(x_2) \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad g(y_1) \leq g(y_2) \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 2.13 [12] An element $(x, y) \in X \times X$ is called a coupled coincidence point of a mapping $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ if $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Definition 2.14 [12] Let X be a nonempty set and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$. We say F and g are commutative if $g(F(x, y)) = F(g(x), g(y))$ for all $x, y \in X$.

Now, we give the notion of an F^* -invariant set and an (F^*, g) -invariant set which is useful for our main results.

Definition 2.15 Let (X, d) be a metric space and $F : X \times X \rightarrow X$ be mapping. Let M be a nonempty subset of X^6 . We say that M is an F^* -invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, u, y, v, z, w) \in M \Leftrightarrow (w, z, v, y, u, x) \in M$;
2. $(x, u, y, v, z, w) \in M \Rightarrow (F(x, u), F(u, x), F(y, v), F(v, y), F(z, w), F(w, z)) \in M$.

Definition 2.16 Let (X, d) be a metric space and $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ are given mapping. Let M be a nonempty subset of X^6 . We say that M is an (F^*, g) -invariant subset of X^6 if and only if, for all $x, y, z, u, v, w \in X$,

1. $(x, u, y, v, z, w) \in M \Leftrightarrow (w, z, v, y, u, x) \in M$;
2. $(g(x), g(u), g(y), g(v), g(z), g(w)) \in M \Rightarrow (F(x, u), F(u, x), F(y, v), F(v, y), F(z, w), F(w, z)) \in M$.

Definition 2.17 Let (X, d) be a metric space and M be a subset of X^6 . We say that M satisfies *the transitive property* if and only if, for all $x, y, w, z, a, b, c, d, e, f \in X$,

$$(x, y, w, z, a, b) \in M \quad \text{and} \quad (a, b, c, d, e, f) \in M \rightarrow (x, y, w, z, e, f) \in M.$$

Remark

1. The set $M = X^6$ is trivially (F^*, g) -invariant, which satisfies the transitive property.
2. Every F^* -invariant set is (F^*, I_X) -invariant when I_X denote identity map on X .

Example 2.18 Let (X, \leq) be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ be a mapping satisfying the mixed g -monotone property. Define a subset $M \subseteq X^6$ by $M = \{(a, b, c, d, e, f) \in X^6, a \geq c \geq e, b \leq d \leq f\}$. Then M is an (F^*, g) -invariant subset of X^6 , which satisfies the transitive property.

Example 2.19 Let $X = R$ and $F : X \times X \rightarrow X$ be defined by $F(x, y) = 1 - x^2$. Let $g : X \rightarrow X$ be given by $g(x) = x - 1$. Then it is easy to show that $M = \{(x, 0, 0, 0, 0, w) \in X^6 : x = w\}$ is an (F^*, g) -invariant subset of X^6 but not an F^* -invariant subset of X^6 as $(1, 0, 0, 0, 0, 1) \in M$ but $(F(1, 0), F(0, 1), F(0, 0), F(0, 0), F(0, 1), F(1, 0)) = (0, 1, 1, 1, 1, 0) \notin M$.

Let Φ denote the set of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ satisfying

1. $\phi^{-1}(\{0\}) = \{0\}$,
2. $\phi(t) < t$ for all $t > 0$,
3. $\lim_{r \rightarrow t^+} \phi(r) < t$ for all $t > 0$.

Lemma 2.20 [12] Let $\phi \in \Phi$. For all $t > 0$, we have $\lim_{n \rightarrow \infty} \phi^n(t) = 0$.

Karapinar *et al.* [34] proved the following theorem.

Theorem 2.21 [34] Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \rightarrow X$, and $g : X \rightarrow X$ such that

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \geq g(y) \geq g(z)$ and $g(u) \leq g(v) \leq g(w)$.

Suppose also that F is continuous and has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Definition 2.22 [34] Let (X, \leq) be a partially ordered set and G be a G -metric on X . We say that (X, G, \leq) is regular if the following conditions hold:

1. if a non-decreasing sequence $(x_n) \rightarrow x$, then $x_n \leq x$ for all n ,
2. if a non-increasing sequence $(y_n) \rightarrow y$, then $y \leq y_n$ for all n .

Theorem 2.23 [34] Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G, \leq) is regular. Suppose that there exist $\phi \in \Phi$, $F : X \times X \rightarrow X$, and $g : X \rightarrow X$ such that

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \geq g(y) \geq g(z)$ and $g(u) \leq g(v) \leq g(w)$.

Suppose also that $(g(X), G)$ is complete, F has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

The purpose of this paper is to present some coupled coincidence point theorems without a mixed g -monotone, using the concept of (F^*, g) -invariant set in complete metric space which are generalizations of the results of Karapinar *et al.* [34].

3 Main results

Theorem 3.1 Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space and M be a nonempty subset of X^6 . Assume that there exists $\phi \in \Phi$ and, also, suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned} \tag{1}$$

for all $(g(x), g(u), g(y), g(v), g(z), g(w)) \in M$.

Suppose also that F is continuous, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F . If there exist $x_0, y_0 \in X \times X$ such that

$$(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)) \in M$$

and M is an (F^*, g) -invariant set which satisfies the transitive property. Then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof Let $(x_0, y_0) \in X \times X$. Since $F(X \times X) \subseteq G(X)$, we can choose $x_1, y_1 \in X$ such that

$$g(x_1) = F(x_0, y_0) \quad \text{and} \quad g(y_1) = F(y_0, x_0).$$

Again from $F(X \times X) \subseteq G(X)$ we can choose $x_2, y_2 \in X$ such that

$$g(x_2) = F(x_1, y_1) \quad \text{and} \quad g(y_2) = F(y_1, x_1).$$

Continuing this process we can construct sequences $\{g(x_n)\}$ and $\{g(y_n)\}$ in X such that

$$g(x_n) = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad g(y_n) = F(y_{n-1}, x_{n-1}) \quad \text{for all } n \geq 1. \tag{2}$$

If there exists $k \in N$ such that $(g(x_{k+1}), g(y_{k+1})) = (g(x_k), g(y_k))$ then $g(x_k) = g(x_{k+1}) = F(x_k, y_k)$ and $g(y_k) = g(y_{k+1}) = F(y_k, x_k)$. Thus, (x_k, y_k) is a coupled coincidence point of F . The proof is completed.

Now we assume that $(g(x_{k+1}), g(y_{k+1})) \neq (g(x_k), g(y_k))$ for all $n \geq 0$. Thus, we have either $g(x_{n+1}) = F(x_n, y_n) \neq g(x_n)$ or $g(y_{n+1}) = F(y_n, x_n) \neq g(y_n)$ for all $n \geq 0$. Since

$$\begin{aligned} &(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)) \\ &= (g(x_1), g(y_1), g(x_1), g(y_1), g(x_0), g(y_0)) \in M \end{aligned}$$

and M is an (F^*, g) -invariant set, we have

$$\begin{aligned} &(F(x_1, y_1), F(y_1, x_1), F(x_1, y_1), F(y_1, x_1), F(x_0, y_0), F(y_0, x_0)) \\ &= (g(x_2), g(y_2), g(x_2), g(y_2), g(x_1), g(y_1)) \in M. \end{aligned}$$

Again, using the fact that M is an (F^*, g) -invariant set, we have

$$\begin{aligned} &(F(x_2, y_2), F(y_2, x_2), F(x_2, y_2), F(y_2, x_2), F(x_1, y_1), F(y_1, x_1)) \\ &= (g(x_3), g(y_3), g(x_3), g(y_3), g(x_2), g(y_2)) \in M. \end{aligned}$$

By repeating this argument, we get

$$\begin{aligned} &(F(x_n, y_n), F(y_n, x_n), F(x_n, y_n), F(y_n, x_n), F(x_{n-1}, y_{n-1}), F(y_{n-1}, x_{n-1})) \\ &= (g(x_n), g(y_n), g(x_n), g(y_n), g(x_{n-1}), g(y_{n-1})) \in M. \end{aligned} \tag{3}$$

From (1), (2), and (3), we have

$$\begin{aligned} &[G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n))] \\ &= G(F(x_n, y_n), F(x_n, y_n), F(x_{n-1}, y_{n-1})) \\ &\quad + G(F(y_n, x_n), F(y_n, x_n), F(y_{n-1}, x_{n-1})) \\ &\leq \phi(G(g(x_n), g(x_n), g(x_{n-1})) + G(g(y_n), g(y_n), g(y_{n-1}))). \end{aligned} \tag{4}$$

Let

$$t_n = G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n)). \tag{5}$$

This implies that

$$t_n \leq \phi(t_{n-1}). \tag{6}$$

Since $\phi(t) < t$ for all $t > 0$, it follows that $\{t_n\}$ is decreasing sequence. Therefore, there is some $\delta \geq 0$ such that $\lim_{n \rightarrow \infty} t_n = \delta$.

We shall prove that $\delta = 0$. Assume, to the contrary, that $\delta > 0$. Then by letting $n \rightarrow \infty$ in (6) and using the properties of the map ϕ , we get

$$\delta = \lim_{n \rightarrow \infty} t_n \leq \lim_{n \rightarrow \infty} \phi(t_{n-1}) = \lim_{t_{n-1} \rightarrow \delta^+} \phi(t_{n-1}) < \delta.$$

This is a contradiction. Thus $\delta = 0$ and hence

$$\lim_{n \rightarrow \infty} t_n = \lim_{n \rightarrow \infty} [G(g(x_{n+1}), g(x_{n+1}), g(x_n)) + G(g(y_{n+1}), g(y_{n+1}), g(y_n))] = 0. \tag{7}$$

Next, we prove that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the G -metric space (X, G) . Suppose, to the contrary, that the least of $\{g(x_n)\}$ and $\{g(y_n)\}$ is not a Cauchy sequence in (X, G) . Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{g(x_{m(k)})\}$ and $\{g(x_{n(k)})\}$ of $\{g(x_n)\}$, $\{g(y_{m(k)})\}$ and $\{g(y_{n(k)})\}$ of $\{g(y_n)\}$ with $m(k) > n(k) \geq K$ such that

$$G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)})) \geq \varepsilon. \tag{8}$$

Further, corresponding to $n(k)$, we can choose $m(k)$ in such a way that it is the smallest integer with $m(k) > n(k) \geq K$ and satisfying (8). Then

$$G(g(x_{m(k)-1}), g(x_{m(k)-1}), g(x_{n(k)})) + G(g(y_{m(k)-1}), g(y_{m(k)-1}), g(y_{n(k)})) < \varepsilon. \tag{9}$$

Using the rectangle inequality, we get

$$\begin{aligned} \varepsilon &\leq r_k \\ &:= G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)})) \\ &\leq G(g(x_{m(k)}), g(x_{m(k)}), g(x_{m(k)-1})) + G(g(x_{m(k)-1}), g(x_{m(k)-1}), g(x_{n(k)})) \\ &\quad + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{m(k)-1})) + G(g(y_{m(k)-1}), g(y_{m(k)-1}), g(y_{n(k)})) \\ &< t_{m(k)-1} + \varepsilon. \end{aligned} \tag{10}$$

Letting $k \rightarrow +\infty$ and using (7), we get

$$\lim_{k \rightarrow \infty} r_k = \lim_{k \rightarrow +\infty} G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)})) = \varepsilon. \tag{11}$$

Again, by the rectangle inequality, we have

$$\begin{aligned} r_k &:= G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)})) \\ &\leq t_{n(k)} + G(g(x_{m(k)}), g(x_{m(k)}), g(x_{m(k)+1})) \\ &\quad + G(g(x_{m(k)+1}), g(x_{m(k)+1}), g(x_{n(k)+1})) \\ &\quad + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{m(k)+1})) + G(g(y_{m(k)+1}), g(y_{m(k)+1}), g(y_{n(k)+1})). \end{aligned}$$

Using the fact that $G(x, x, y) \leq 2G(x, y, y)$ for any $x, y \in X$, we obtain

$$\begin{aligned} r_k &\leq t_{n(k)} + 2t_{m(k)} + G(g(x_{m(k)+1}), g(x_{m(k)+1}), g(x_{n(k)+1})) \\ &\quad + G(g(y_{m(k)+1}), g(y_{m(k)+1}), g(y_{n(k)+1})). \end{aligned}$$

Since $m(k) > n(k)$,

$$(g(x_{m(k)}), g(y_{m(k)}), g(x_{m(k)}), g(y_{m(k)}), g(x_{m(k)-1}), g(y_{m(k)-1})) \in M$$

and

$$(g(x_{m(k)-1}), g(y_{m(k)-1}), g(x_{m(k)-1}), g(y_{m(k)-1}), g(x_{m(k)-2}), g(y_{m(k)-2})) \in M.$$

From M being an (F^*, g) -invariant set which satisfies the transitive property, we have

$$(g(x_{m(k)}), g(y_{m(k)}), g(x_{m(k)}), g(y_{m(k)}), g(x_{m(k)-2}), g(y_{m(k)-2})) \in M.$$

Again from

$$(g(x_{m(k)-2}), g(y_{m(k)-2}), g(x_{m(k)-2}), g(y_{m(k)-2}), g(x_{m(k)-3}), g(y_{m(k)-3})) \in M$$

we get

$$(g(x_{m(k)}), g(y_{m(k)}), g(x_{m(k)}), g(y_{m(k)}), g(x_{n(k)}), g(y_{n(k)})) \in M.$$

Now, using (1), we have

$$\begin{aligned} &G(g(x_{m(k)+1}), g(x_{m(k)+1}), g(x_{n(k)+1})) + G(g(y_{m(k)+1}), g(y_{m(k)+1}), g(y_{n(k)+1})) \\ &= G(F(x_{m(k)}, y_{m(k)}), F(x_{m(k)}, y_{m(k)}), F(x_{n(k)}, y_{n(k)})) \\ &\quad + G(F(y_{m(k)}, x_{m(k)}), F(y_{m(k)}, x_{m(k)}), F(y_{n(k)}, x_{n(k)})) \\ &\leq \phi(G(g(x_{m(k)}), g(x_{m(k)}), g(x_{n(k)})) + G(g(y_{m(k)}), g(y_{m(k)}), g(y_{n(k)}))) \\ &\leq \phi(r_k). \end{aligned} \tag{12}$$

Letting $k \rightarrow +\infty$ in (12) and using (7) and (11) and $\lim_{r \rightarrow t^+} \phi(r) < t$ for all $t > 0$, we have

$$\varepsilon = \lim_{k \rightarrow \infty} r_k \leq \lim_{n \rightarrow \infty} \phi(r_k) = \lim_{r_k \rightarrow \varepsilon^+} \phi(r_k) < \varepsilon.$$

This is a contradiction. This shows that $\{g(x_n)\}$ and $\{g(y_n)\}$ are Cauchy sequences in the G -metric space (X, G) . Since (X, G) is complete, $\{g(x_n)\}$ and $\{g(y_n)\}$ are G -convergent; there exist $x, y \in X$ such that $\lim_{n \rightarrow \infty} g(x_n) = x$ and $\lim_{n \rightarrow \infty} g(y_n) = y$. That is, from Proposition 2.4, we have

$$\lim_{n \rightarrow \infty} G(g(x_n), g(x_n), x) = \lim_{n \rightarrow \infty} G(g(x_n), x, x) = 0, \tag{13}$$

$$\lim_{n \rightarrow \infty} G(g(y_n), g(y_n), y) = \lim_{n \rightarrow \infty} G(g(y_n), y, y) = 0. \tag{14}$$

From (13), (14), continuity of g , and Proposition 2.7, we get

$$\lim_{n \rightarrow \infty} G(g(g(x_n)), g(g(x_n), g(x))) = \lim_{n \rightarrow \infty} G(g(g(x_n)), g(x), g(x)) = 0, \tag{15}$$

$$\lim_{n \rightarrow \infty} G(g(g(y_n)), g(g(y_n), g(y))) = \lim_{n \rightarrow \infty} G(g(g(y_n)), g(y), g(y)) = 0. \tag{16}$$

From (2) and commutativity of F and g ,

$$g(g(x_{n+1})) = g(F(x_n, y_n)) = F(g(x_n), g(y_n)), \tag{17}$$

$$g(g(y_{n+1})) = g(F(y_n, x_n)) = F(g(y_n), g(x_n)). \tag{18}$$

We now show that $F(x, y) = g(x)$ and $F(y, x) = g(y)$.

Taking the limit as $n \rightarrow +\infty$ in (17) and (18), by (15), (16), continuity of F , and commutativity of F and g , we get

$$\begin{aligned} g(x) &= g\left(\lim_{n \rightarrow \infty} g(x_{n+1})\right) = \lim_{n \rightarrow \infty} g(g(x_{n+1})) = \lim_{n \rightarrow \infty} g(F(x_n, y_n)) \\ &= \lim_{n \rightarrow \infty} F(g(x_n), g(y_n)) = F(x, y) \end{aligned}$$

and

$$\begin{aligned} g(y) &= g\left(\lim_{n \rightarrow \infty} g(y_{n+1})\right) = \lim_{n \rightarrow \infty} g(g(y_{n+1})) = \lim_{n \rightarrow \infty} g(F(y_n, x_n)) \\ &= \lim_{n \rightarrow \infty} F(g(y_n), g(x_n)) = F(y, x). \end{aligned}$$

Thus we prove that $F(x, y) = g(x)$ and $F(y, x) = g(y)$. □

Theorem 3.2 *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space and M be a nonempty subset of X^6 . Assume that there exists $\phi \in \Phi$ and, also, suppose that $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that*

$$\begin{aligned} &[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ &\leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned}$$

for all $(g(x), g(u), g(y), g(v), g(z), g(w)) \in M$.

Suppose also that $(g(X), G)$ is complete $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F and if any two sequences $\{x_n\}, \{y_n\}$ with $(x_{n+1}, y_{n+1}, x_{n+1}, y_{n+1}, x_n, y_n) \in M, \{x_n\} \rightarrow x$ and $\{y_n\} \rightarrow y$ for all $n \geq 1$, then $(x, y, x_n, y_n, x_n, y_n) \in M$. If there exist $(x_0, y_0) \in X \times X$ such that $(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)) \in M$ and M is an (F^*, g) -invariant set which satisfies the transitive property. Then there exist $x, y \in X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof Consider a Cauchy sequences $\{g(x_n)\}, \{g(y_n)\}$ as in the proof of Theorem 3.1. Since $(g(X), G)$ is a complete metric space, there exists $x, y \in X$ such that $\{g(x_n)\} \rightarrow g(x)$ and $\{g(y_n)\} \rightarrow g(y)$ by the assumption, and we have $(g(x), g(y), g(x_n), g(y_n), g(x_n), g(y_n)) \in M$ for all $n \geq 1$; by the rectangle inequality, (1), and $\phi(t) < t$ for all $t > 0$, we get

$$\begin{aligned} &G(F(x, y), g(x), g(x)) + G(F(y, x), g(y), g(y)) \\ &\leq G(F(x, y), g(x_{n+1}), g(x_{n+1})) + G(g(x_{n+1}), g(x), g(x)) \\ &\quad + G(F(y, x), g(y_{n+1}), g(y_{n+1})) + G(g(y_{n+1}), g(y), g(y)) \\ &= G(F(x, y), F(x_n, y_n), F(x_n, y_n)) + G(g(x_{n+1}), g(x), g(x)) \end{aligned}$$

$$\begin{aligned}
 &+ G(F(y, x), F(y_n, x_n), F(y_n, x_n)) + G(g(y_{n+1}), g(y), g(y)) \\
 \leq &\phi(G(g(x), g(x_n), g(x_n)) + G(g(y), g(y_n), g(y_n))) \\
 &+ G(g(x_{n+1}), g(x), g(x)) + G(g(y_{n+1}), g(y), g(y)) \\
 < &G(g(x), g(x_n), g(x_n)) + G(g(y), g(y_n), g(y_n)) \\
 &+ G(g(x_{n+1}), g(x), g(x)) + G(g(y_{n+1}), g(y), g(y)).
 \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ in the above inequality, we obtain

$$G(F(x, y), g(x), g(x)) + G(F(y, x), g(y), g(y)) = 0.$$

This implies that $g(x) = F(x, y)$ and $g(y) = F(y, x)$. □

Example 3.3 Let $X = \mathbb{R}$. Define $G : X \times X \times X \rightarrow [0, +\infty)$ by $G(x, y, z) = |x - y| + |x - z| + |y - z|$ and let $F : X \times X \rightarrow X$ be defined by

$$F(x, y) = \frac{x + 2y}{4}, \quad (x, y) \in X^2,$$

and $g : X \rightarrow X$ by $g(x) = \frac{3x}{2}$. Let $y_1 = 2$ and $y_2 = 4$. Then we have $g(y_1) \leq g(y_2)$, but $F(x, y_1) \leq F(x, y_2)$, and so the mapping F does not satisfy the mixed g -monotone property.

Letting $x, u, y, v, z, w \in X$, we have

$$\begin{aligned}
 &[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, u), F(w, z))] \\
 &= \left| \frac{x + 2u}{4} - \frac{y + 2v}{4} \right| + \left| \frac{x + 2u}{4} - \frac{z + 2w}{4} \right| + \left| \frac{y + 2v}{4} - \frac{z + 2w}{4} \right| \\
 &\quad + \left| \frac{u + 2x}{4} - \frac{v + 2y}{4} \right| + \left| \frac{u + 2x}{4} - \frac{w + 2z}{4} \right| + \left| \frac{v + 2y}{4} - \frac{w + 2z}{4} \right| \\
 &\leq 3 \left| \frac{x - y}{4} \right| + 3 \left| \frac{x - z}{4} \right| + 3 \left| \frac{y - z}{4} \right| + 3 \left| \frac{u - v}{4} \right| + 3 \left| \frac{u - w}{4} \right| + 3 \left| \frac{v - w}{4} \right| \\
 &= \frac{3}{4} (|x - y| + |x - z| + |y - z|) + \frac{3}{4} (|u - v| + |u - w| + |v - w|)
 \end{aligned}$$

and we have

$$\begin{aligned}
 &G(g(x), g(y), g(z)) + G(g(u), g(v), g(w)) \\
 &= G\left(\frac{3x}{2}, \frac{3y}{2}, \frac{3z}{2}\right) + G\left(\frac{3u}{2}, \frac{3v}{2}, \frac{3w}{2}\right) \\
 &= \frac{3}{2} (|x - y| + |x - z| + |y - z|) + \frac{3}{2} (|u - v| + |u - w| + |v - w|).
 \end{aligned}$$

Put $\phi(t) = t/2$, then

$$\begin{aligned}
 &[G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, u), F(w, z))] \\
 &\leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))).
 \end{aligned}$$

If we apply Theorem 3.1 with $M = X^6$ then F satisfy (1). So by our theorem we see that F has a coupled coincidence point $(0, 0)$.

Remark Although the mixed monotone property is an essential tool in the partially ordered G -metric spaces to show the existence of coupled coincidence points, the mappings do not have the mixed g -monotone property in the general case as in the above example. Therefore, Theorem 3.1 and Theorem 3.2 are interesting, as a new auxiliary tool, in showing the existence of a coupled coincidence point.

Theorem 3.4 *In addition to the hypotheses of Theorem 3.1, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exist $(u, v) \in X \times X$ such that*

$$(g(u), g(v), g(x), g(y), g(x), g(y)) \in M \quad \text{and} \quad (g(u), g(v), g(x^*), g(y^*), g(x^*), g(y^*)) \in M.$$

Suppose also that ϕ is a non-decreasing function. Then F and g have a unique coupled common fixed point, that is, there exist unique $(x, y) \in X \times X$ such that $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.

Proof From Theorem 3.1 the set of coupled coincidence point is nonempty. Suppose (x, y) and (x^*, y^*) are coupled coincidence point of F , that is,

$$g(x) = F(x, y), \quad g(y) = F(y, x), \quad g(x^*) = F(x^*, y^*) \quad \text{and} \quad g(y^*) = F(y^*, x^*).$$

We shall show that

$$g(x^*) = g(x) \quad \text{and} \quad g(y^*) = g(y). \tag{19}$$

By assumption there is $(u, v) \in X \times X$ such that

$$(g(u), g(v), g(x), g(y), g(x), g(y)) \in M \quad \text{and} \quad (g(u), g(v), g(x^*), g(y^*), g(x^*), g(y^*)) \in M.$$

Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ such that $g(u_1) = F(u_0, v_0)$ and $g(v_1) = F(v_0, u_0)$. Then similarly as in Theorem 3.1, we can inductively define sequences $\{g(u_n)\}$ and $\{g(v_n)\}$ such that

$$g(u_n) = F(u_{n-1}, v_{n-1}) \quad \text{and} \quad g(v_n) = F(v_{n-1}, u_{n-1}) \quad \text{for all } n \geq 1.$$

Since M is (F^*, g) -invariant and $(g(u_0), g(v_0), g(x), g(y), g(x), g(y)) \in M$, we have

$$(F(u_0, v_0), F(v_0, u_0), F(x, y), F(y, x), F(x, y), F(y, x)) \in M.$$

That is $(g(u_1), g(v_1), g(x), g(y), g(x), g(y)) \in M$.

From $(g(u_1), g(v_1), g(x), g(y), g(x), g(y)) \in M$, if we use again the property of (F^*, g) -invariance, then it follows that

$$(F(u_1, v_1), F(v_1, u_1), F(x, y), F(y, x), F(x, y), F(y, x)) \in M$$

and so $(g(u_2), g(v_2), g(x), g(y), g(x), g(y)) \in M$.

By repeating this process, we get

$$(g(u_n), g(v_n), g(x), g(y), g(x), g(y)) \in M \quad \text{for all } n \geq 1. \tag{20}$$

Thus from (1) and (20), we have

$$\begin{aligned} &G(g(u_{n+1}), g(x), g(x)) + G(g(v_{n+1}), g(y), g(y)) \\ &= G(F(u_n, v_n), F(x, y), F(x, y)) + G(F(v_n, u_n), F(y, x), F(y, x)) \\ &\leq \phi(G(g(u_n), g(x), g(x)) + G(g(v_n), g(y), g(y))). \end{aligned} \tag{21}$$

Since ϕ is non-decreasing from (21), we get

$$\begin{aligned} &G(g(u_{n+1}), g(x), g(x)) + G(g(v_{n+1}), g(y), g(y)) \\ &\leq \phi^n(G(g(u_1), g(x), g(x)) + G(g(v_1), g(y), g(y))). \end{aligned} \tag{22}$$

This holds for each $n \geq 1$. Letting $n \rightarrow +\infty$ in (22), using Lemma 2.20 implies

$$\lim_{n \rightarrow \infty} G(g(u_{n+1}), g(x), g(x)) = \lim_{n \rightarrow \infty} G(g(v_{n+1}), g(y), g(y)) = 0. \tag{23}$$

Similarly, we obtain

$$\lim_{n \rightarrow \infty} G(g(u_{n+1}), g(x^*), g(x^*)) = \lim_{n \rightarrow \infty} G(g(v_{n+1}), g(y^*), g(y^*)) = 0. \tag{24}$$

Hence, from (23), (24), and Proposition 2.4, we get $g(x^*) = g(x)$ and $g(y^*) = g(y)$.

Since $g(x) = F(x, y)$ and $g(y) = F(y, x)$, by commutativity of F and g , we have

$$g(g(x)) = g(F(x, y)) = F(g(x), g(y)) \quad \text{and} \quad g(g(y)) = g(F(y, x)) = F(g(y), g(x)). \tag{25}$$

Denote $g(x) = z$ and $g(y) = w$. Then from (25)

$$g(z) = F(z, w) \quad \text{and} \quad g(w) = F(w, z). \tag{26}$$

Therefore, (z, w) is a coupled coincidence fixed point of F and g . Then from (19) with $x^* = z$ and $y^* = w$, it follows that $g(z) = g(x)$ and $g(w) = g(y)$, that is,

$$g(z) = z \quad \text{and} \quad g(w) = w. \tag{27}$$

From (26) and (27), $z = g(z) = F(z, w)$ and $w = g(w) = F(w, z)$. Therefore, (z, w) is a coupled common fixed point of F and g .

To prove the uniqueness, assume that (p, q) is another coupled common fixed point. Then by (19) we have $p = g(p) = g(z) = z$ and $q = g(q) = g(w) = w$. □

Next, we give a simple application of our results to coupled coincidence point theorems in partially ordered metric spaces.

Corollary 3.5 *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G) is a complete G -metric space. Suppose that there exist $\phi \in \Phi$, $F : X \times X \rightarrow X$, and $g : X \rightarrow X$ such that*

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \geq g(y) \geq g(z)$ and $g(u) \leq g(v) \leq g(w)$.

Suppose also that F is continuous and has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$ and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof We define the subset $M \subseteq X^6$ by

$$M = \{(x, u, y, v, z, w) \in X^6 : x \geq y \geq z, u \leq v \leq w\}.$$

From Example 2.18, M is an (F^*, g) -invariant set which satisfies the transitive property. By (1), we have

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, y), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))). \end{aligned}$$

Since $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0).$$

We have $(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)) \in M$ because F is continuous. By Theorem 3.1, we have $x = F(x, y)$ and $y = F(y, x)$. □

Corollary 3.6 *Let (X, \leq) be a partially ordered set and G be a G -metric on X such that (X, G, \leq) is regular. Suppose that there exists $\phi \in \Phi$, $F : X \times X \rightarrow X$ and $g : X \rightarrow X$ such that*

$$\begin{aligned} & [G(F(x, u), F(y, v), F(z, w)) + G(F(u, x), F(v, u), F(w, z))] \\ & \leq \phi(G(g(x), g(y), g(z)) + G(g(u), g(v), g(w))) \end{aligned}$$

for all $x, y, z, u, v, w \in X$ for which $g(x) \geq g(y) \geq g(z)$ and $g(u) \leq g(v) \leq g(w)$.

Suppose also that $(g(X), G)$ is complete, F has the mixed g -monotone property, $F(X \times X) \subseteq G(X)$, and g is continuous and commutes with F . If there exist $x_0, y_0 \in X$ such that

$$g(x_0) \leq F(x_0, y_0) \quad \text{and} \quad g(y_0) \geq F(y_0, x_0),$$

then there exist $(x, y) \in X \times X$ such that $g(x) = F(x, y)$ and $g(y) = F(y, x)$.

Proof As in Corollary 3.5, we get

$$(F(x_0, y_0), F(y_0, x_0), F(x_0, y_0), F(y_0, x_0), g(x_0), g(y_0)) \in M.$$

For any two sequences $\{g(x_n)\}$, $\{g(y_n)\}$ such that $\{g(x_n)\}$ is non-decreasing sequence $\{g(x_n)\} \rightarrow g(x)$ and $\{g(y_n)\}$ is non-increasing sequence $\{g(y_n)\} \rightarrow g(y)$. We have

$$g(x_1) \leq g(x_2) \leq \dots \leq g(x_n) \leq g(x)$$

and

$$g(y_1) \geq g(y_2) \geq \dots \geq g(y_n) \geq g(y) \quad \text{for all } n \geq 1.$$

Therefore, we have $(g(x), g(y), g(x_n), g(y_n), g(x_n), g(y_n)) \in M$ for all $n \geq 1$. So the assumption of Theorem 3.2 holds and hence F has a coupled coincidence point. \square

Corollary 3.7 *In addition to the hypothesis of Corollary 3.5, suppose that for every $(x, y), (x^*, y^*) \in X \times X$ there exists a $(u, v) \in X \times X$ such that $(F(u, v), F(v, u))$ is comparable to $(F(x, y), F(y, x))$ and $(F(x^*, y^*), F(y^*, x^*))$. Suppose also that ϕ is a non-decreasing function. Then F and g have a unique coupled common fixed point, that is, there exists a unique $(x, y) \in X \times X$ such that $x = g(x) = F(x, y)$ and $y = g(y) = F(y, x)$.*

Proof We define the subset $M \subseteq X^6$ by $M = \{(x, u, y, v, z, w) \in X^6 : x \geq y \geq z, u \leq v \leq w\}$. From Example 2.18, M is an (F^*, g) -invariant set which satisfies the transitive property. Thus, the proof of the existence of a coupled fixed point is straightforward by following the same lines as in the proof of Corollary 3.5.

Next, we show the uniqueness of a coupled fixed point of F .

Since for all $(x, y), (x^*, y^*) \in X \times X$, there exist $(u, v) \in X \times X$ such that $g(x) \leq g(u), g(y) \geq g(v)$ and $g(x^*) \leq g(u), g(y^*) \geq g(v)$, we can conclude that

$$(g(u), g(v), g(x), g(y), g(x), g(y)) \in M$$

and

$$(g(u), g(v), g(x^*), g(y^*), g(x^*), g(y^*)) \in M.$$

Therefore, since all the hypotheses of Theorem 3.4 hold F has a unique coupled fixed point. The proof is completed. \square

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors read and approved the final manuscript.

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