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Sharp maximal function inequalities and boundedness for Toeplitz type operator associated to singular integral operator with non-smooth kernel

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Abstract

In this paper, we establish the sharp maximal function inequalities for the Toeplitz type operator associated to some singular integral operator with non-smooth kernel. As an application, we obtain the boundedness of the operator on Morrey and Triebel-Lizorkin spaces.

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1 Introduction and preliminaries

As the development of the singular integral operators, their commutators and multilinear operators have been well studied (see [1–3]). In [1, 2], the authors prove that the commutators generated by the singular integral operators and *BMO* functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo (see [4]) proves a similar result when singular integral operators are replaced by the fractional integral operators. In [5, 6], the boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel-Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained. In [7–9], some Toeplitz type operators associated to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by *BMO* and Lipschitz functions are obtained. In [10, 11], some singular integral operators with non-smooth kernel are introduced, and the boundedness for the operators and their commutators are obtained (see [9, 12–16]). The main purpose of this paper is to study the Toeplitz type operator generated by the singular integral operator with non-smooth kernel and the Lipschitz and *BMO* functions.

Definition 1 A family of operators D_t, $t > 0$ is said to be an 'approximation to the identity' if, for every $t > 0$, D_t can be represented by a kernel $a_t(x, y)$ in the following sense:

$$D_t(f)(x) = \int_{R^n} a_t(x, y)f(y)\,dy$$

for every $f \in L^p(R^n)$ with $p \geq 1$, and $a_t(x,y)$ satisfies

$$\left|a_t(x,y)\right| \leq h_t(x,y) = Ct^{-n/2}\rho\left(|x-y|^2/t\right),$$

where ρ is a positive, bounded, and decreasing function satisfying

$$\lim_{r\to\infty} r^{n+\epsilon}\rho\left(r^2\right) = 0$$

for some $\epsilon > 0$.

Definition 2 A linear operator T is called a singular integral operator with non-smooth kernel if T is bounded on $L^2(R^n)$ and associated with a kernel $K(x,y)$ such that

$$T(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$$

for every continuous function f with compact support, and for almost all x not in the support of f.

(1) There exists an 'approximation to the identity' $\{B_t, t > 0\}$ such that TB_t has the associated kernel $k_t(x,y)$ and there exist $c_1, c_2 > 0$ so that

$$\int_{|x-y|>c_1t^{1/2}} \left|K(x,y) - k_t(x,y)\right|dx \leq c_2 \quad \text{for all } y \in R^n.$$

(2) There exists an 'approximation to the identity' $\{A_t, t > 0\}$ such that A_tT has the associated kernel $K_t(x,y)$ which satisfies

$$\left|K_t(x,y)\right| \leq c_4t^{-n/2} \quad \text{if } |x-y| \leq c_3t^{1/2},$$

and

$$\left|K(x,y) - K_t(x,y)\right| \leq c_4t^{\delta/2}|x-y|^{-n-\delta} \quad \text{if } |x-y| \geq c_3t^{1/2},$$

for some $\delta > 0$, $c_3, c_4 > 0$.

Let b be a locally integrable function on R^n and T be the singular integral operator with non-smooth kernel. The Toeplitz type operator associated to T is defined by

$$T_b = \sum_{k=1}^{m}\left(T^{k,1}M_bI_\alpha T^{k,2} + T^{k,3}I_\alpha M_bT^{k,4}\right),$$

where $T^{k,1}$ are the singular integral operator with non-smooth kernel T or $\pm I$ (the identity operator), $T^{k,2}$ and $T^{k,4}$ are the linear operators, $T^{k,3} = \pm I$, $k = 1,\dots,m$, $M_b(f) = bf$ and I_α is the fractional integral operator ($0 < \alpha < n$) (see [4]).

Note that the commutator $[b,T](f) = bT(f) - T(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operator T_b is the non-trivial generalizations of the commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [2]). In [10, 11], the boundedness of the singular integral operator with non-smooth kernel are obtained. In [12–16], the boundedness of the commutator associated to the singular integral operator with non-smooth kernel are obtained. Our works is motivated by these papers. In this paper, we will prove the sharp maximal inequalities for the Toeplitz type operator T_b. As the application, we obtain the Morrey and Triebel-Lizorkin spaces boundedness for the Toeplitz type operator T_b.

Definition 3 Let $0 < \beta < 1$ and $1 \leq p < \infty$. The Triebel-Lizorkin space associated with the 'approximations to the identity' $\{A_t, t > 0\}$ is defined by

$$\dot{F}_{p,A}^{\beta,\infty}(R^n) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{\dot{F}_{p,A}^{\beta,\infty}} < \infty\},$$

where

$$\|f\|_{\dot{F}_{p,A}^{\beta,\infty}} = \left\| \sup_{Q \ni \cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x) - A_{t_Q}(f)(x)| \, dx \right\|_{L^p},$$

and the supremum is taken over all cubes Q of R^n with sides parallel to the axes, $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

Now, let us introduce some notations. Throughout this paper, $Q = Q(x, r)$ will denote a cube of R^n with sides parallel to the axes and center at x and edge is r. For any locally integrable function f, the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy,$$

where, and in what follows $f_Q = |Q|^{-1} \int_Q f(x) \, dx$. It is well known that (see [3])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| \, dy$$

and

$$\|b - b_{2^k Q}\|_{BMO} \leq Ck\|b\|_{BMO} \quad \text{for } k \geq 1.$$

We say that f belongs to $BMO(R^n)$ if $f^{\#}$ belongs to $L^\infty(R^n)$ and $\|f\|_{BMO} = \|f^{\#}\|_{L^\infty}$.

Let M be the Hardy-Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} |Q|^{-1} \int_Q |f(y)| \, dy.$$

For $\eta > 0$, let $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$.

For $0 \leq \eta < n$ and $1 \leq r < \infty$, set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1-r\eta/n}} \int_Q |f(y)|^r \, dy \right)^{1/r}.$$

The A_1 weight is defined by (see [17])

$$A_1 = \{w \in L^p_{\text{loc}}(R^n) : M(w)(x) \leq Cw(x), \text{a.e.}\}.$$

The sharp maximal function $M_A(f)$ associated with the 'approximation to the identity' $\{A_t, t > 0\}$ is defined by

$$M^\#_A(f)(x) = \sup_{x\in Q} \frac{1}{|Q|} \int_Q |f(y) - A_{t_Q}(f)(y)|\, dy,$$

where $t_Q = l(Q)^2$ and $l(Q)$ denotes the side length of Q.

For $\beta > 0$, the Lipschitz space $\text{Lip}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\text{Lip}_\beta} = \sup_{\substack{x,y\in R^n\\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\beta} < \infty.$$

Throughout this paper, φ will denote a positive, increasing function on R^+ for which there exists a constant $D > 0$ such that

$$\varphi(2t) \leq D\varphi(t) \quad \text{for } t \geq 0.$$

Let f be a locally integrable function on R^n. Set, for $0 \leq \eta < n$ and $1 \leq p < n/\eta$,

$$\|f\|_{L^{p,\eta,\varphi}} = \sup_{x\in R^n, d>0} \left(\frac{1}{\varphi(d)^{1-p\eta/n}} \int_{Q(x,d)} |f(y)|^p\, dy \right)^{1/p}.$$

The generalized fractional Morrey spaces are defined by

$$L^{p,\eta,\varphi}(R^n) = \{f \in L^1_{\text{loc}}(R^n) : \|f\|_{L^{p,\eta,\varphi}} < \infty\}.$$

We write $L^{p,\eta,\varphi}(R^n) = L^{p,\varphi}(R^n)$ if $\eta = 0$, which is the generalized Morrey space. If $\varphi(d) = d^\delta$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$, which is the classical Morrey space (see [18, 19]). As the Morrey space may be considered as an extension of the Lebesgue space (the Morrey space $L^{p,\lambda}$ becomes the Lebesgue space L^p when $\lambda = 0$), it is natural and important to study the boundedness of the operator on the Morrey spaces $L^{p,\lambda}$ with $\lambda > 0$ (see [20–23]). The purpose of this paper is twofold. First, we establish some sharp inequalities for the Toeplitz type operator T_b, and, second, we prove the boundedness for the Toeplitz type operator by using the sharp inequalities.

2 Theorems and lemmas

We shall prove the following theorems.

Theorem 1 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $0 < \beta < 1$, $1 < s < \infty$ *and* $b \in \text{Lip}_\beta(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C^\infty_0(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^\#_A(T_b(f))(\tilde{x}) \leq C\|b\|_{\text{Lip}_\beta} \sum_{k=1}^m \big(M_{\beta,s}(I_\alpha T^{k,2}(f))(\tilde{x}) + M_{\beta+\alpha,s}(T^{k,4}(f))(\tilde{x})\big).$$

Theorem 2 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $0 < \beta < \min(1, \delta)$, $1 < s < \infty$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left|T_b(f)(x) - A_{t_Q}\left(T_b(f)\right)(x)\right| dx \leq C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m \left(M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}) + M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x})\right).$$

Theorem 3 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $1 < s < \infty$ *and* $b \in BMO(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C_0^\infty(R^n)$ *and* $\tilde{x} \in R^n$,

$$M_A^\#\left(T_b(f)\right)(\tilde{x}) \leq C\|b\|_{BMO} \sum_{k=1}^m \left(M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}) + M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x})\right).$$

Theorem 4 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $0 < \beta < 1$, $1 < p < n/(\alpha + \beta)$, $1/q = 1/p - (\alpha + \beta)/n$, $0 < D < 2^n$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$ *and* $T^{k,2}$ *and* $T^{k,4}$ *are the bounded operators on* $L^{p,\varphi}(R^n)$ *for* $1 < p < \infty$, $k = 1, \ldots, m$, *then* T_b *is bounded from* $L^{p,\alpha+\beta,\varphi}(R^n)$ *to* $L^{q,\varphi}(R^n)$.

Theorem 5 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $0 < \beta < \min(1, \epsilon)$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$ *and* $T^{k,2}$ *and* $T^{k,4}$ *are the bounded operators on* $L^p(R^n)$ *for* $1 < p < \infty$, $k = 1, \ldots, m$, *then* T_b *is bounded from* $L^p(R^n)$ *to* $\dot{F}_{q,A}^{\beta,\infty}(R^n)$.

Theorem 6 *Let T be the singular integral operator with non-smooth kernel as Definition* 2, $0 < D < 2^n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $b \in BMO(R^n)$. *If* $T_1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$ *and* $T^{k,2}$ *and* $T^{k,4}$ *are the bounded operators on* $L^{p,\varphi}(R^n)$ *for* $1 < p < \infty$, $k = 1, \ldots, m$, *then* T_b *is bounded from* $L^{p,\alpha,\varphi}(R^n)$ *to* $L^{q,\varphi}(R^n)$.

Corollary 1 *Let* $[b, T](f) = bT(f) - T(bf)$ *be the commutator generated by the singular integral operator T with non-smooth kernel and b. Then Theorems* 1-6 *hold for* $[b, T]$.

To prove the theorems, we need the following lemmas.

Lemma 1 ([10, 11]) *Let T be the singular integral operator with non-smooth kernel as Definition* 2. *Then, for every* $f \in L^p(R^n)$, $1 < p < \infty$,

$$\left\|T(f)\right\|_{L^p} \leq C\|f\|_{L^p}.$$

Lemma 2 ([10, 11]) *Let* $\{A_t, t > 0\}$ *be an 'approximation to the identity'. For any* $\gamma > 0$, *there exists a constant* $C > 0$ *independent of* γ *such that*

$$\left|\left\{x \in R^n : M(f)(x) > D\lambda, M_A^\#(f)(x) \leq \gamma\lambda\right\}\right| \leq C\gamma\left|\left\{x \in R^n : M(f)(x) > \lambda\right\}\right|$$

for $\lambda > 0$, *where D is a fixed constant which only depends on n. Thus, for* $f \in L^p(R^n)$, $1 < p < \infty$ *and* $w \in A_1$,

$$\|M(f)\|_{L^p(w)} \le C\|M_A^{\#}(f)\|_{L^p(w)}.$$

Lemma 3 (See [14]) *Let* $\{A_t, t > 0\}$ *be an 'approximation to the identity' and* $\tilde{K}_{\alpha,t}(x,y)$ *be the kernel of difference operator* $I_\alpha - A_t I_\alpha$. *Then*

$$|\tilde{K}_{\alpha,t}(x,y)| \le C\frac{t}{|x-y|^{n+2-\alpha}}.$$

Lemma 4 (See [4, 17]) *Suppose that* $0 \le \alpha < n$, $1 \le s < p < n/\alpha$, $1/q = 1/p - \alpha/n$ *and* $w \in A_1$. *Then*

$$\|I_\alpha(f)\|_{L^q(w)} \le C\|f\|_{L^p(w)}$$

and

$$\|M_{\alpha,s}(f)\|_{L^q(w)} \le C\|f\|_{L^p(w)}.$$

Lemma 5 *Let* $\{A_t, t > 0\}$ *be an 'approximation to the identity' and* $0 < D < 2^n$. *Then*

(a) $\|M(f)\|_{L^{p,\varphi}} \le C\|M_A^{\#}(f)\|_{L^{p,\varphi}}$ *for* $1 < p < \infty$;

(b) $\|I_\alpha(f)\|_{L^{q,\varphi}} \le C\|f\|_{L^{p,\alpha,\varphi}}$ *for* $0 < \alpha < n$, $1 < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$;

(c) $\|M_{\alpha,s}(f)\|_{L^{q,\varphi}} \le C\|f\|_{L^{p,\alpha,\varphi}}$ *for* $0 \le \alpha < n$, $1 \le s < p < n/\alpha$ *and* $1/q = 1/p - \alpha/n$.

Proof (a) For any cube $Q = Q(x_0, d)$ in R^n, we know $M(\chi_Q) \in A_1$ for any cube $Q = Q(x,d)$ by [24]. Noticing that $M(\chi_Q) \le 1$ and $M(\chi_Q)(x) \le d^n/(|x-x_0|-d)^n$ if $x \in Q^c$, by Lemma 2, we have

$$\begin{aligned}
\int_Q M(f)(x)^p\,dx &= \int_{R^n} M(f)(x)^p \chi_Q(x)\,dx \\
&\le \int_{R^n} M(f)(x)^p M(\chi_Q)(x)\,dx \\
&\le C\int_{R^n} M_A^{\#}(f)(x)^p M(\chi_Q)(x)\,dx \\
&= C\biggl(\int_Q M_A^{\#}(f)(x)^p M(\chi_Q)(x)\,dx \\
&\quad + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M_A^{\#}(f)(x)^p M(\chi_Q)(x)\,dx\biggr) \\
&\le C\biggl(\int_Q M_A^{\#}(f)(x)^p\,dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q\setminus 2^kQ} M_A^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|}\,dx\biggr) \\
&\le C\biggl(\int_Q M_A^{\#}(f)(x)^p\,dx + \sum_{k=0}^{\infty}\int_{2^{k+1}Q} M_A^{\#}(f)(x)^p 2^{-kn}\,dy\biggr) \\
&\le C\|M_A^{\#}(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d)
\end{aligned}$$

$$\begin{aligned}&\le C\|M_A^{\#}(f)\|_{L^{p,\varphi}}^p\sum_{k=0}^{\infty}\left(2^{-n}D\right)^k\varphi(d)\\&\le C\|M_A^{\#}(f)\|_{L^{p,\varphi}}^p\varphi(d),\end{aligned}$$

thus

$$\|M(f)\|_{L^{p,\varphi}}\le C\|M_A^{\#}(f)\|_{L^{p,\varphi}}.$$

The proofs of (b) and (c) are similar to that of (a) by Lemma 4, we omit the details. □

3 Proofs of theorems

Proof of Theorem 1 It suffices to prove for $f\in C_0^\infty(R^n)$, the following inequality holds:

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|T_b(f)(x)-A_{t_Q}\left(T_b(f)\right)(x)\right|dx\\&\quad\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m\left(M_{\beta,s}\left(I_\alpha T^{k,2}(f)\right)(\tilde{x})+M_{\beta+\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x})\right),\end{aligned}$$

where $t_Q=(l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are T $(k=1,\dots,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. We write, by $T_1(g)=0$,

$$\begin{aligned}T_b(f)(x)&=\sum_{k=1}^m T^{k,1}M_bI_\alpha T^{k,2}(f)(x)+\sum_{k=1}^m T^{k,3}I_\alpha M_bT^{k,4}(f)(x)\\&=U_b(x)+V_b(x)=U_{b-b_{2Q}}(x)+V_{b-b_{2Q}}(x),\end{aligned}$$

where

$$\begin{aligned}U_{b-b_{2Q}}(x)&=\sum_{k=1}^m T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)+\sum_{k=1}^m T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x)\\&=U_1(x)+U_2(x)\end{aligned}$$

and

$$\begin{aligned}V_{b-b_{2Q}}(x)&=\sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)+\sum_{k=1}^m T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,4}(f)(x)\\&=V_1(x)+V_2(x).\end{aligned}$$

Then

$$\begin{aligned}&\frac{1}{|Q|}\int_Q\left|T_b(f)(x)-A_{t_Q}\left(T_b(f)\right)(x)\right|dx\\&\quad\le\frac{1}{|Q|}\int_Q\left|U_1(x)\right|dx+\frac{1}{|Q|}\int_Q\left|V_1(x)\right|dx\\&\qquad+\frac{1}{|Q|}\int_Q\left|A_{t_Q}(U_1)(x)\right|dx+\frac{1}{|Q|}\int_Q\left|A_{t_Q}(V_1)(x)\right|dx\end{aligned}$$

$$+\frac{1}{|Q|}\int_Q\left|U_2(x)-A_{t_Q}(U_2)(x)\right|dx+\frac{1}{|Q|}\int_Q\left|V_2(x)-A_{t_Q}(V_2)(x)\right|dx$$
$$=I_1+I_2+I_3+I_4+I_5+I_6.$$

Now, let us estimate I_1, I_2, I_3, I_4, I_5, and I_6, respectively. For I_1, by Hölder's inequality and Lemma 1, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|dx\\
&\quad\le\left(\frac{1}{|Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}\\
&\quad\le C|Q|^{-1/s}\left(\int_{R^n}\left|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}\\
&\quad\le C|Q|^{-1/s}\left(\int_{2Q}\left(\left|b(x)-b_{2Q}\right|\left|I_\alpha T^{k,2}(f)(x)\right|\right)^s dx\right)^{1/s}\\
&\quad\le C|Q|^{-1/s}\|b\|_{\mathrm{Lip}_\beta}|2Q|^{\beta/n}|2Q|^{1/s-\beta/n}\left(\frac{1}{|2Q|^{1-s\beta/n}}\int_{2Q}\left|I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}\\
&\quad\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_1&\le\sum_{k=1}^m\frac{1}{|Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|dx\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\beta,s}\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

For I_2, by Lemma 4, we obtain, for $1/r=1/s-\alpha/n$,

$$\begin{aligned}
I_2&\le\sum_{k=1}^m\left(\frac{1}{|Q|}\int_{R^n}\left|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)\right|^r dx\right)^{1/r}\\
&\le C\sum_{k=1}^m|Q|^{-1/r}\left(\int_{2Q}\left(\left|b(x)-b_{2Q}\right|\left|T^{k,4}(f)(x)\right|\right)^s dx\right)^{1/s}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m|Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n}\left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}\left|T^{k,4}(f)(x)\right|^s dx\right)^{1/s}\\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\beta+\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}).
\end{aligned}$$

For I_3, by the condition on h_{t_Q} and notice for $x\in Q$, $y\in 2^{j+1}Q\backslash 2^jQ$, then $h_{t_Q}(x,y)\le Ct_Q^{-n/2}\rho(2^{2(j-1)})$, we obtain, similar to the proof of I_1,

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q\left|A_{t_Q}\left(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)\right)(x)\right|dx\\
&\quad\le\frac{C}{|Q|}\int_Q\int_{R^n}h_{t_Q}(x,y)\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|dy\,dx
\end{aligned}$$

$$\begin{aligned}
&\le \frac{C}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|\,dy\,dx \\
&\quad+\frac{C}{|Q|}\int_Q\int_{(2Q)^c} h_{t_Q}(x,y)\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|\,dy\,dx \\
&\le \frac{C}{|Q|}\int_Q\int_{2Q} t_Q^{-n/2}\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|\,dy\,dx \\
&\quad+C\sum_{j=1}^\infty t_Q^{-n/2}\rho\bigl(2^{2(j-1)}\bigr)\bigl(2^j l(Q)\bigr)^n\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q\setminus 2^jQ}\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|\,dy \\
&\le \frac{C}{|Q|}\int_{2Q}\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|\,dy \\
&\quad+C\sum_{j=1}^\infty 2^{jn}\rho\bigl(2^{2(j-1)}\bigr)\biggl(\frac{1}{|2^{j+1}Q|}\int_{R^n}\bigl|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|^s\,dy\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}\bigl(I_\alpha T^{k,2}(f)\bigr)(\tilde{x}) \\
&\quad+C\sum_{j=1}^\infty 2^{jn}\rho\bigl(2^{2(j-1)}\bigr)\biggl(\frac{1}{|2^{j+1}Q|}\int_{R^n}\bigl|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\bigr|^s\,dy\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}\bigl(I_\alpha T^{k,2}(f)\bigr)(\tilde{x}) \\
&\quad+C\sum_{j=1}^\infty 2^{(j-1)(n+\epsilon)}\rho\bigl(2^{2(j-1)}\bigr)2^{-j(\epsilon+n/s)}\biggl(\frac{1}{|2Q|^{1/s-\beta/n}}\int_{2Q}\bigl|I_\alpha T^{k,2}(f)(y)\bigr|^s\,dy\biggr)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}\bigl(I_\alpha T^{k,2}(f)\bigr)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_3 &\le \sum_{k=1}^m\frac{1}{|Q|}\int_{R^n}\bigl|A_{t_Q}\bigl(T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)\bigr)(x)\bigr|\,dx \\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\beta,s}\bigl(I_\alpha T^{k,2}(f)\bigr)(\tilde{x}).
\end{aligned}$$

Similarly, by Lemma 4, for $1/r = 1/s - \alpha/n$,

$$\begin{aligned}
I_4 &\le \sum_{k=1}^m\frac{1}{|Q|}\int_Q\bigl|A_{t_Q}\bigl(T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)\bigr)(x)\bigr|\,dx \\
&\le \sum_{k=1}^m\frac{C}{|Q|}\int_Q\int_{2Q} h_{t_Q}(x,y)\bigl|T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\bigr|\,dy\,dx \\
&\quad+\sum_{k=1}^m\frac{C}{|Q|}\int_Q\int_{(2Q)^c} h_{t_Q}(x,y)\bigl|T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\bigr|\,dy\,dx \\
&\le C\sum_{k=1}^m t_Q^{-n/2}|Q|^{1-1/r}\biggl(\int_{R^n}\bigl|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\bigr|^r\,dy\biggr)^{1/r} \\
&\quad+C\sum_{k=1}^m\sum_{j=1}^\infty t_Q^{-n/2}\rho\bigl(2^{2(j-1)}\bigr)|Q|^{1-1/r}\biggl(\int_{R^n}\bigl|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\bigr|^r\,dy\biggr)^{1/r}
\end{aligned}$$

$$\begin{aligned}
&\leq C\sum_{k=1}^{m}|Q|^{-1/r}\left(\int_{2Q}\left|\left(b(y)-b_{2Q}\right)T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s}\\
&\quad+C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\rho\left(2^{2(j-1)}\right)|Q|^{-1/r}\left(\int_{2Q}\left|\left(b(y)-b_{2Q}\right)T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s}\\
&\leq C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}|Q|^{-1/r}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\alpha)/n}\left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}\left|T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s}\\
&\quad+C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho\left(2^{2(j-1)}\right)2^{-j(n+\epsilon)}\left(\frac{1}{|2Q|^{1-s(\beta+\alpha)/n}}\int_{2Q}\left|T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s}\\
&\leq C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\beta+\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}).
\end{aligned}$$

For I_5, we get, for $x\in Q$,

$$\begin{aligned}
&\left|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x)-A_{t_Q}\left(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)\right)(x)\right|\\
&\quad\leq\int_{(2Q)^c}\left|b(y)-b_{2Q}\right|\left|K(x-y)-K_{t_Q}(x-y)\right|\left|I_\alpha T^{k,2}(f)(y)\right|dy\\
&\quad\leq C\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\|b\|_{\mathrm{Lip}_\beta}\left|2^{j+1}Q\right|^{\beta/n}\frac{l(Q)^\delta}{|x_0-y|^{n+\delta}}\left|I_\alpha T^{k,2}(f)(y)\right|dy\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta}\sum_{j=1}^{\infty}2^{-j\delta}\left(\frac{1}{|2^{j+1}Q|^{1-s\beta/n}}\int_{2^{j+1}Q}\left|I_\alpha T^{k,2}(f)(y)\right|^{s}dy\right)^{1/s}\\
&\quad\leq C\|b\|_{\mathrm{Lip}_\beta}M_{\beta,s}\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_5&\leq\frac{1}{|Q|}\int_Q\sum_{k=1}^{m}\left|T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)(x)-A_{t_Q}\left(T^{k,1}M_{(b-b_{2Q})\chi_{(2Q)^c}}I_\alpha T^{k,2}(f)\right)(x)\right|dx\\
&\leq C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\beta,s}\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

Similarly, by Lemma 3, we get

$$\begin{aligned}
I_6&\leq\frac{1}{|Q|}\int_Q\sum_{k=1}^{m}\left|T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,4}(f)(x)\right.\\
&\quad\left.-A_{t_Q}\left(T^{k,3}I_\alpha M_{(b-b_{2Q})\chi_{(2Q)^c}}T^{k,4}(f)\right)(x)\right|dx\\
&\leq C\sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\int_{(2Q)^c}\left|b(y)-b_{2Q}\right|\left|\tilde{K}_{t_Q}(x-y)\right|\left|T^{k,4}(f)(y)\right|dy\,dx\\
&\leq C\sum_{k=1}^{m}\frac{1}{|Q|}\sum_{j=1}^{\infty}\int_{2^jd\leq|y-x_0|<2^{j+1}d}\|b\|_{\mathrm{Lip}_\beta}\left(2^{j+1}d\right)^{\beta}\frac{t_Q}{|x-y|^{n+2-\alpha}}\left|T^{k,4}(f)(y)\right|dy
\end{aligned}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m\sum_{j=1}^\infty 2^{-2j}\left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\alpha)/n}}\int_{2^{j+1}Q}\left|T^{k,4}(f)(y)\right|^s dy\right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\beta+\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}).$$

These complete the proof of Theorem 1. □

Proof of Theorem 2 It suffices to prove for $f\in C_0^\infty(R^n)$, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_b(f)(x)-A_{t_Q}\left(T_b(f)\right)(x)\right|dx$$
$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m\left(M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x})+M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x})\right),$$

where $t_Q=(l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are T ($k=1,\ldots,m$). Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. Similar to the proof of Theorem 1, we have

$$\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|T_b(f)(x)-A_{t_Q}\left(T_b(f)\right)(x)\right|dx$$
$$\le\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|U_1(x)\right|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|V_1(x)\right|dx$$
$$+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|A_{t_Q}(U_1)(x)\right|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|A_{t_Q}(V_1)(x)\right|dx$$
$$+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|U_2(x)-A_{t_Q}(U_2)(x)\right|dx+\frac{1}{|Q|^{1+\beta/n}}\int_Q\left|V_2(x)-A_{t_Q}(V_2)(x)\right|dx$$
$$=J_1+J_2+J_3+J_4+J_5+J_6.$$

By using the same argument as in the proof of Theorem 1, we get

$$J_1\le\sum_{k=1}^m|Q|^{-\beta/n}\left(\frac{1}{|Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}$$
$$\le C\sum_{k=1}^m|Q|^{-\beta/n-1/s}\left(\int_{R^n}\left|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}$$
$$\le C\sum_{k=1}^m|Q|^{-\beta/n-1/s}\|b\|_{\mathrm{Lip}_\beta}|2Q|^{\beta/n}|2Q|^{1/s}\left(\frac{1}{|2Q|}\int_{2Q}\left|I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}$$
$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}),$$
$$J_2\le\sum_{k=1}^m|Q|^{-\beta/n}\left(\frac{1}{|Q|}\int_{R^n}\left|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)\right|^r dx\right)^{1/r}$$
$$\le C\sum_{k=1}^m|Q|^{-\beta/n-1/r}\left(\int_{2Q}\left(\left|b(x)-b_{2Q}\right|\left|T^{k,4}(f)(x)\right|\right)^s dx\right)^{1/s}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}|2Q|^{\beta/n}|2Q|^{1/s-\alpha/n}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\left|T^{k,4}(f)(x)\right|^{s}dx\right)^{1/s}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}),$$

$$\begin{aligned}
J_3 &\le C\sum_{k=1}^{m}|Q|^{-1-\beta/n}\int_Q\int_{2Q}t_Q^{-n/2}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|dy\,dx \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}|Q|^{-\beta/n}2^{jn}\rho\left(2^{2(j-1)}\right) \\
&\quad \times\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\le \sum_{k=1}^{m}\frac{C}{|Q|^{1+\beta/n}}\int_{2Q}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|dy \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}|Q|^{-\beta/n}2^{jn}\rho\left(2^{2(j-1)}\right)\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}\left|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}) \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho\left(2^{2(j-1)}\right)2^{-j(\epsilon+n/s)}\left(\frac{1}{|2Q|}\int_{2Q}\left|I_\alpha T^{k,2}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
J_4 &\le C\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}\left(\int_{R^n}\left|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\right|^{r}dy\right)^{1/r} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\rho\left(2^{2(j-1)}\right)|Q|^{-1/r}\left(\int_{R^n}\left|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\right|^{r}dy\right)^{1/r} \\
&\le C\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}\left(\int_{2Q}\left|\left(b(y)-b_{2Q}\right)T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\rho\left(2^{2(j-1)}\right)|Q|^{-1/r}\left(\int_{2Q}\left|\left(b(y)-b_{2Q}\right)T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}|Q|^{-\beta/n-1/r}|2Q|^{\beta/n}|2Q|^{1/s-\alpha/n}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\left|T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{(j-1)(n+\epsilon)}\rho\left(2^{2(j-1)}\right)2^{-j(n+\epsilon)}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\left|T^{k,4}(f)(y)\right|^{s}dy\right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}),
\end{aligned}$$

$$
\begin{aligned}
J_5 &\le \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{k=1}^m \int_{(2Q)^c} |b(y) - b_{2Q}| |K(x-y) - K_{t_Q}(x-y)| |I_\alpha T^{k,2}(f)(y)| \, dy \\
&\le C \sum_{k=1}^m |Q|^{-\beta/n} \sum_{j=1}^\infty \int_{2^j d \le |y-x_0| < 2^{j+1} d} \|b\|_{\mathrm{Lip}_\beta} |2^{j+1}Q|^{\beta/n} \frac{l(Q)^\delta}{|x_0 - y|^{n+\delta}} |I_\alpha T^{k,2}(f)(y)| \, dy \\
&\le C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m \sum_{j=1}^\infty 2^{j(\beta-\delta)} \left(\frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |I_\alpha T^{k,2}(f)(y)|^s \, dy \right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x}), \\
J_6 &\le \sum_{k=1}^m \frac{1}{|Q|^{1+\beta/n}} \int_Q \int_{(2Q)^c} |b(y) - b_{2Q}| |\tilde{K}_{t_Q}(x-y)| |T^{k,4}(f)(y)| \, dy \, dx \\
&\le C \sum_{k=1}^m \frac{1}{|Q|^{\beta/n}} \sum_{j=1}^\infty \int_{2^j d \le |y-x_0| < 2^{j+1} d} \|b\|_{\mathrm{Lip}_\beta} \big(2^{j+1}d\big)^\beta \frac{t_Q}{|x-y|^{n+2-\alpha}} |T^{k,4}(f)(y)| \, dy \\
&\le C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m \sum_{j=1}^\infty 2^{j(\beta-2)} \left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}} \int_{2^{j+1}Q} |T^{k,4}(f)(y)|^s \, dy \right)^{1/s} \\
&\le C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\alpha,s}\big(T^{k,4}(f)\big)(\tilde{x}).
\end{aligned}
$$

These complete the proof of Theorem 2. □

Proof of Theorem 3 It suffices to prove for $f \in C_0^\infty(R^n)$, the following inequality holds:

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q |T_b(f)(x) - A_{t_Q}\big(T_b(f)\big)(x)| \, dx \\
&\quad \le C\|b\|_{BMO} \sum_{k=1}^m \Big(M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x}) + M_{\alpha,s}\big(T^{k,4}(f)\big)(\tilde{x}) \Big),
\end{aligned}
$$

where $t_Q = (l(Q))^2$ and $l(Q)$ denotes the side length of Q. Without loss of generality, we may assume $T^{k,1}$ are T $(k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Similar to the proof of Theorem 1, we have

$$
\begin{aligned}
&\frac{1}{|Q|} \int_Q |T_b(f)(x) - A_{t_Q}\big(T_b(f)\big)(x)| \, dx \\
&\quad \le \frac{1}{|Q|} \int_Q |U_1(x)| \, dx + \frac{1}{|Q|} \int_Q |V_1(x)| \, dx \\
&\qquad + \frac{1}{|Q|} \int_Q |A_{t_Q}(U_1)(x)| \, dx + \frac{1}{|Q|} \int_Q |A_{t_Q}(V_1)(x)| \, dx \\
&\qquad + \frac{1}{|Q|} \int_Q |U_2(x) - A_{t_Q}(U_2)(x)| \, dx + \frac{1}{|Q|} \int_Q |V_2(x) - A_{t_Q}(V_2)(x)| \, dx \\
&\quad = L_1 + L_2 + L_3 + L_4 + L_5 + L_6.
\end{aligned}
$$

Now, let us estimate L_1, L_2, L_3, L_4, L_5 and L_6, respectively. For L_1, we obtain, for $1 < r < s$,

$$\begin{aligned}
L_1 &\le \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(x)\right|^r dx\right)^{1/r} \\
&\le C\sum_{k=1}^{m}|Q|^{-1/r}\left(\int_{2Q}\left(\left|b(x)-b_{2Q}\right|\left|I_\alpha T^{k,2}(f)(x)\right|\right)^r dx\right)^{1/r} \\
&\le C\sum_{k=1}^{m}\left(\frac{1}{|2Q|}\int_{2Q}\left|I_\alpha T^{k,2}(f)(x)\right|^s dx\right)^{1/s}\left(\frac{1}{|2Q|}\int_{2Q}\left|b(x)-b_{2Q}\right|^{sr/(s-r)} dx\right)^{(s-r)/sr} \\
&\le C\|b\|_{BMO}\sum_{k=1}^{m}M_s\left(I_\alpha T^{k,2}(f)\right)(\tilde{x}).
\end{aligned}$$

For L_2, we obtain, for $1 < v < s$ and $1/v = 1/u - \alpha/n$,

$$\begin{aligned}
L_2 &\le \sum_{k=1}^{m}\left(\frac{1}{|Q|}\int_{R^n}\left|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(x)\right|^v dx\right)^{1/v} \\
&\le C\sum_{k=1}^{m}|Q|^{-1/v}\left(\int_{2Q}\left(\left|b(x)-b_{2Q}\right|\left|T^{k,4}(f)(x)\right|\right)^u dx\right)^{1/u} \\
&\le C\sum_{k=1}^{m}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\left|T^{k,4}(f)(x)\right|^s dx\right)^{1/s}\left(\frac{1}{|2Q|}\int_{2Q}\left|b(x)-b_{2Q}\right|^{su/(s-u)} dx\right)^{(s-u)/su} \\
&\le C\|b\|_{BMO}\sum_{k=1}^{m}M_{\alpha,s}\left(T^{k,4}(f)\right)(\tilde{x}).
\end{aligned}$$

For L_3, by $h_{t_Q}(x,y) \le Ct_Q^{-n/2}\rho(2^{2(j-1)})$ for $x \in Q$, $y \in 2^{j+1}Q\backslash 2^jQ$, we obtain, for $1 < r < s$,

$$\begin{aligned}
L_3 &\le \sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\int_{2Q}h_{t_Q}(x,y)\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right| dy\,dx \\
&\quad + \sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\int_{(2Q)^c}h_{t_Q}(x,y)\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right| dy\,dx \\
&\le \sum_{k=1}^{m}\frac{C}{|Q|}\int_Q\int_{2Q}t_Q^{-n/2}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right| dy\,dx \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}t_Q^{-n/2}\rho\left(2^{2(j-1)}\right)\left(2^jl(Q)\right)^n \\
&\quad \times\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|^r dy\right)^{1/r} \\
&\le \sum_{k=1}^{m}\frac{C}{|Q|}\int_{2Q}\left|T^{k,1}M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right| dy \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}2^{jn}\rho\left(2^{2(j-1)}\right)\left(\frac{1}{|2^{j+1}Q|}\int_{R^n}\left|M_{(b-b_{2Q})\chi_{2Q}}I_\alpha T^{k,2}(f)(y)\right|^r dy\right)^{1/r}
\end{aligned}$$

$$\begin{aligned}
&\leq C\|b\|_{BMO}\sum_{k=1}^{m} M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x}) \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty} 2^{(j-1)(n+\epsilon)}\rho\big(2^{2(j-1)}\big)2^{-j(\epsilon+n/r)}\left(\frac{1}{|2Q|}\int_{2Q}\big|I_\alpha T^{k,2}(f)(y)\big|^s\,dy\right)^{1/s} \\
&\quad \times \left(\frac{1}{|2Q|}\int_{2Q}\big|b(y)-b_{2Q}\big|^{sr/(s-r)}\,dy\right)^{(s-r)/sr} \\
&\leq C\|b\|_{BMO}\sum_{k=1}^{m} M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x}).
\end{aligned}$$

Similarly, for $1/s + 1/s' = 1$, $1 < v < s$ and $1/v = 1/u - \alpha/n$, we get

$$\begin{aligned}
L_4 &\leq C\sum_{k=1}^{m} t_Q^{-n/2}|Q|^{1-1/v}\left(\int_{R^n}\big|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\big|^v\,dy\right)^{1/v} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty} t_Q^{-n/2}\rho\big(2^{2(j-1)}\big)|Q|^{1-1/v}\left(\int_{R^n}\big|I_\alpha M_{(b-b_{2Q})\chi_{2Q}}T^{k,4}(f)(y)\big|^v\,dy\right)^{1/v} \\
&\leq C\sum_{k=1}^{m}|Q|^{-1/v}\left(\int_{2Q}\big|\big(b(y)-b_{2Q}\big)T^{k,4}(f)(y)\big|^u\,dy\right)^{1/u} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\rho\big(2^{2(j-1)}\big)|Q|^{-1/v}\left(\int_{2Q}\big|\big(b(y)-b_{2Q}\big)T^{k,4}(f)(y)\big|^u\,dy\right)^{1/u} \\
&\leq C\sum_{k=1}^{m}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\big|T^{k,4}(f)(y)\big|^s\,dy\right)^{1/s} \\
&\quad \times \left(\frac{1}{|2Q|}\int_{2Q}\big|b(y)-b_{2Q}\big|^{su/(s-u)}\,dy\right)^{(s-u)/su} \\
&\quad + C\sum_{k=1}^{m}\sum_{j=1}^{\infty} 2^{(j-1)(n+\epsilon)}\rho\big(2^{2(j-1)}\big)2^{-j(n+\epsilon)}\left(\frac{1}{|2Q|^{1-s\alpha/n}}\int_{2Q}\big|T^{k,4}(f)(y)\big|^s\,dy\right)^{1/s} \\
&\quad \times \left(\frac{1}{|2Q|}\int_{2Q}\big|b(y)-b_{2Q}\big|^{su/(s-u)}\,dy\right)^{(s-u)/su} \\
&\leq C\|b\|_{BMO}\sum_{k=1}^{m} M_{\alpha,s}\big(T^{k,4}(f)\big)(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
L_5 &\leq \frac{1}{|Q|}\int_Q\sum_{k=1}^{m}\int_{(2Q)^c}\big|b(y)-b_{2Q}\big|\big|K(x-y)-K_{t_Q}(x-y)\big|\big|I_\alpha T^{k,2}(f)(y)\big|\,dy\,dx \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^j d\leq|y-x_0|<2^{j+1}d}\big|b(y)-b_{2Q}\big|\frac{l(Q)^\delta}{|x_0-y|^{n+\delta}}\big|I_\alpha T^{k,2}(f)(y)\big|\,dy \\
&\leq C\sum_{k=1}^{m}\sum_{j=1}^{\infty} 2^{-j\delta}\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\big|b(y)-b_{2Q}\big|^{s'}\,dy\right)^{1/s'} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\big|I_\alpha T^{k,2}(f)(y)\big|^s\,dy\right)^{1/s}
\end{aligned}$$

$$\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty} j2^{-j\delta}\|b\|_{BMO} M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x})$$

$$\le C\|b\|_{BMO}\sum_{k=1}^{m} M_s\big(I_\alpha T^{k,2}(f)\big)(\tilde{x}),$$

$$L_6 \le \sum_{k=1}^{m}\frac{1}{|Q|}\int_Q\int_{(2Q)^c}\big|b(y)-b_{2Q}\big|\big|\tilde{K}_{t_Q}(x-y)\big|\big|T^{k,4}(f)(y)\big|\,dy\,dx$$

$$\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty}\int_{2^j d\le|y-x_0|<2^{j+1}d}\big|b(y)-b_{2Q}\big|\frac{t_Q}{|x-y|^{n+2-\alpha}}\big|T^{k,4}(f)(y)\big|\,dy$$

$$\le C\sum_{k=1}^{m}\sum_{j=1}^{\infty} j2^{-2j}\left(\frac{1}{|2^{j+1}Q|^{1-s\alpha/n}}\int_{2^{j+1}Q}\big|T^{k,4}(f)(y)\big|^s\,dy\right)^{1/s}$$

$$\times\left(\frac{1}{|2^{j+1}Q|}\int_{2^{j+1}Q}\big|b(y)-b_{2Q}\big|^{s'}\,dy\right)^{1/s'}$$

$$\le C\|b\|_{BMO}\sum_{k=1}^{m} M_{\alpha,s}\big(T^{k,4}(f)\big)(\tilde{x}).$$

These complete the proof of Theorem 3. □

Proof of Theorem 4 Choose $1 < s < p$ in Theorem 1 and set $1/r = 1/p - \alpha/n$. We have, by Lemma 5,

$$\big\|T_b(f)\big\|_{L^{q,\varphi}} \le \big\|M\big(T_b(f)\big)\big\|_{L^{q,\varphi}} \le C\big\|M_A^{\#}\big(T_b(f)\big)\big\|_{L^{q,\varphi}}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\Big(\big\|M_{\beta,s}\big(I_\alpha T^{k,2}(f)\big)\big\|_{L^{q,\varphi}} + \big\|M_{\beta+\alpha,s}\big(T^{k,4}(f)\big)\big\|_{L^{q,\varphi}}\Big)$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\Big(\big\|I_\alpha T^{k,2}(f)\big\|_{L^{r,\beta,\varphi}} + \big\|T^{k,4}(f)\big\|_{L^{p,\alpha+\beta,\varphi}}\Big)$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\Big(\big\|T^{k,2}(f)\big\|_{L^{p,\alpha+\beta,\varphi}} + \|f\|_{L^{p,\alpha+\beta,\varphi}}\Big)$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p,\alpha+\beta,\varphi}}.$$

This completes the proof of the theorem. □

Proof of Theorem 5 Choose $1 < s < p$ in Theorem 2. We have, by Lemma 4,

$$\big\|T_b(f)\big\|_{\dot{F}^{\beta,\infty}_{q,A}} \le C\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q\big|T_b(f)(x)-A_{t_Q}\big(T_b(f)\big)(x)\big|\,dx\right\|_{L^q}$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\Big(\big\|M_s\big(I_\alpha T^{k,2}(f)\big)\big\|_{L^q} + \big\|M_{\alpha,s}\big(T^{k,4}(f)\big)\big\|_{L^q}\Big)$$

$$\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^{m}\Big(\big\|I_\alpha T^{k,2}(f)\big\|_{L^q} + \big\|T^{k,4}(f)\big\|_{L^p}\Big)$$

$$\begin{aligned}&\le C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m\bigl(\bigl\|T^{k,2}(f)\bigr\|_{L^p}+\|f\|_{L^p}\bigr)\\&\le C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.\end{aligned}$$

This completes the proof of the theorem. □

Proof of Theorem 6 Choose $1 < s < p$ in Theorem 3, we have, by Lemma 5,

$$\begin{aligned}\bigl\|T_b(f)\bigr\|_{L^{q,\varphi}} &\le \bigl\|M\bigl(T_b(f)\bigr)\bigr\|_{L^{q,\varphi}} \le C\bigl\|M_A^{\#}\bigl(T_b(f)\bigr)\bigr\|_{L^{q,\varphi}}\\&\le C\|b\|_{BMO}\sum_{k=1}^m\bigl(\bigl\|M_s\bigl(I_\alpha T^{k,2}(f)\bigr)\bigr\|_{L^{q,\varphi}}+\bigl\|M_{\alpha,s}\bigl(T^{k,4}(f)\bigr)\bigr\|_{L^{q,\varphi}}\bigr)\\&\le C\|b\|_{BMO}\sum_{k=1}^m\bigl(\bigl\|I_\alpha T^{k,2}(f)\bigr\|_{L^{q,\varphi}}+\bigl\|T^{k,4}(f)\bigr\|_{L^{p,\alpha,\varphi}}\bigr)\\&\le C\|b\|_{BMO}\sum_{k=1}^m\bigl(\bigl\|T^{k,2}(f)\bigr\|_{L^{p,\alpha,\varphi}}+\|f\|_{L^{p,\alpha,\varphi}}\bigr)\\&\le C\|b\|_{BMO}\|f\|_{L^{p,\alpha,\varphi}}.\end{aligned}$$

This completes the proof of the theorem. □

4 Applications

In this section we shall apply Theorems 1-6 of the paper to the holomorphic functional calculus of linear elliptic operators. First, we review some definitions regarding the holomorphic functional calculus (see [10, 11]). Given $0 \le \theta < \pi$. Define

$$S_\theta = \bigl\{z \in C : \bigl|\arg(z)\bigr| \le \theta\bigr\} \cup \{0\}$$

and its interior by S_θ^0. Set $\tilde{S}_\theta = S_\theta \setminus \{0\}$. A closed operator L on some Banach space E is said to be of type θ if its spectrum $\sigma(L) \subset S_\theta$ and for every $\nu \in (\theta, \pi]$, there exists a constant C_ν such that

$$|\eta|\bigl\|(\eta I - L)^{-1}\bigr\| \le C_\nu, \quad \eta \notin \tilde{S}_\theta.$$

For $\nu \in (0, \pi]$, let

$$H_\infty\bigl(S_\mu^0\bigr) = \bigl\{f : S_\theta^0 \to C : f \text{ is holomorphic and } \|f\|_{L^\infty} < \infty\bigr\},$$

where $\|f\|_{L^\infty} = \sup\{|f(z)| : z \in S_\mu^0\}$. Set

$$\Psi\bigl(S_\mu^0\bigr) = \left\{g \in H_\infty\bigl(S_\mu^0\bigr) : \exists s > 0, \exists c > 0 \text{ such that } \bigl|g(z)\bigr| \le c\frac{|z|^s}{1+|z|^{2s}}\right\}.$$

If L is of type θ and $g \in H_\infty(S_\mu^0)$, we define $g(L) \in L(E)$ by

$$g(L) = -(2\pi i)^{-1}\int_\Gamma (\eta I - L)^{-1} g(\eta)\, d\eta,$$

where Γ is the contour $\{\xi = re^{\pm i\phi} : r \geq 0\}$ parameterized clockwise around S_θ with $\theta < \phi < \mu$. If, in addition, L is one-one and has dense range, then, for $f \in H_\infty(S_\mu^0)$,

$$f(L) = \left[h(L)\right]^{-1}(fh)(L),$$

where $h(z) = z(1+z)^{-2}$. L is said to have a bounded holomorphic functional calculus on the sector S_μ, if

$$\left\|g(L)\right\| \leq N\|g\|_{L^\infty}$$

for some $N > 0$ and for all $g \in H_\infty(S_\mu^0)$.

Now, let L be a linear operator on $L^2(R^n)$ with $\theta < \pi/2$ so that $(-L)$ generates a holomorphic semigroup e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$. Applying Theorem 6 of [11] and Theorems 1-6, we get

Corollary 2 *Assume the following conditions are satisfied*:

(i) *The holomorphic semigroup* e^{-zL}, $0 \leq |\arg(z)| < \pi/2 - \theta$ *is represented by the kernels* $a_z(x,y)$ *which satisfy, for all* $\nu > \theta$, *an upper bound*

$$\left|a_z(x,y)\right| \leq c_\nu h_{|z|}(x,y)$$

for $x, y \in R^n$, *and* $0 \leq |\arg(z)| < \pi/2 - \theta$, *where* $h_t(x,y) = Ct^{-n/2}s(|x-y|^2/t)$ *and* s *is a positive, bounded, and decreasing function satisfying*

$$\lim_{r\to\infty} r^{n+\epsilon}s\left(r^2\right) = 0.$$

(ii) *The operator* L *has a bounded holomorphic functional calculus in* $L^2(R^n)$, *that is, for all* $\nu > \theta$ *and* $g \in H_\infty(S_\mu^0)$, *the operator* $g(L)$ *satisfies*

$$\left\|g(L)(f)\right\|_{L^2} \leq c_\nu \|g\|_{L^\infty}\|f\|_{L^2}.$$

Let $g(L)_b$ *be the Toeplitz type operator associated to* $g(L)$. *Then Theorems* 1-6 *hold for* $g(L)_b$.

Competing interests
The author declares that they have no competing interests.

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