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On strong law of large numbers and growth rate for a class of random variables

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Abstract

In this paper, we study the strong law of large numbers for a class of random variables satisfying the maximal moment inequality with exponent 2. Our results embrace the Kolmogorov strong law of large numbers and the Marcinkiewicz strong law of large numbers for this class of random variables. In addition, strong growth rate for weighted sums of this class of random variables is presented.

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Keywords: strong law of large numbers; weighted sums; with exponent 2

1 Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. $S_n = \sum_{i=1}^n X_i$, $n \geq 1$, $S_0 = 0$. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of constant with $0 < b_n \uparrow \infty$. Then $\{a_n X_n, n \geq 1\}$ is said to obey the general strong law of large numbers (SLLN) with norming constant $\{b_n, n \geq 1\}$ if the normed weighted sums

$$\frac{1}{b_n}\sum_{i=1}^n a_i(X_i - EX_i) \to 0 \quad \text{almost surely (a.s.)} \tag{1.1}$$

holds. Note that the SLLN of the form (1.1) embraces the Kolmogorov SLLN ($b_n = n, a_n = 1$) and the Marcinkiewicz SLLN ($b_n = n^{1/r}$, $a_n = 1$, $r > 0$). When $b_n = \sum_{i=1}^n a_i$, fundamental results for the SLLN were obtained.

Under an independent assumption, many SLLNs for the weighted sums are obtained. One can refer to Adler and Rosalsky [1], Chow and Teicher [2], Fernholz and Teicher [3], Jamison *et al.* [4] and Teicher [5].

Under a pairwise independent assumption, Rosalsky [6] obtained some SLLNs for weighted sums of pairwise independent and identically distributed random variables. Sung [7] obtained sufficient conditions for (1.1) if $\{X_n, n \geq 1\}$ is a sequence of pairwise independent random variables satisfying $\int_0^\infty x^{p-1} \sup_{n\geq 1} P(|X_n| > x)\,dx < \infty$. Sung [8] presented the following result: $\frac{1}{a_n}\sum_{i=1}^n (X_i - EX_i I(|X_i| \leq a_i)) \to 0$ a.s., where $\{a_n\}$ is a sequence of positive constants with $\frac{a_n}{n} \uparrow$ and $\{X_n\}$ is a sequence of pairwise independent and identically distributed random variables.

For more details about strong limit theorems for dependent case, one can refer to Wu [9], Wu and Jiang [10], Hu *et al.* [11], Shen *et al.* [12], Zhou *et al.* [13] and Zhou [14], and so forth.

Recently Sung [15] gave the following definition.

Definition 1.1 (Sung [15]) A random variable sequence $\{X_n, n \geq 1\}$ is said to satisfy the maximal moment inequality with exponent 2 if for all $n \geq m \geq 1$, there exists a constant C independent of n and m such that

$$E\left(\max_{m\leq k\leq n}\left|\sum_{i=m}^{k} X_i\right|^2\right) \leq C\sum_{i=m}^{n} EX_i^2. \tag{1.2}$$

We can see that a wide class of mean zero random variables satisfies (1.2). Inspired by Sung [7, 15], we establish SLLN of the form (1.1) for a class of random variables satisfying the maximal moment inequality with exponent 2.

The rest of the paper is organized as follows. In Section 2, some preliminary definition and lemmas are presented. In Section 3, main results and their proofs are provided.

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant not depending on n, which may be different in various places. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive numbers, $a_n \ll b_n$ represents that there exists a constant $C > 0$ such that $a_n \leq Cb_n$ for all n.

2 Preliminaries

The following lemmas and definition will be needed in this paper.

Lemma 2.1 (Sung [7]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and put $G(x) = \sup_{n\geq 1} P(|X_n| > x)$ for $x \geq 0$. Assume that $\int_0^\infty x^{p-1}G(x)\,dx < \infty$ for some $1 \leq p < 2$. Then*
(i) $\sum_{n=1}^\infty P(|X_n| > n^{1/p}) < \infty$.
(ii) $\sum_{n=1}^\infty EX_n^2 I(|X_n| \leq n^{1/p})/n^{2/p} < \infty$.
(iii) $EX_nI(|X_n| > c_n) \to 0$ *for any sequence $\{c_n, n \geq 1\}$ satisfying $c_n \to \infty$.*

Lemma 2.2 (Sung [15]) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables satisfying the maximal moment inequality with exponent 2. If $\sum_{n=1}^\infty EX_n^2 < \infty$, then $\sum_{n=1}^\infty X_n$ converges almost surely.*

Definition 2.3 A random variable sequence $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a constant C such that

$$P\big(|X_n| > x\big) \leq CP\big(|X| > x\big)$$

for all $x \geq 0$ and $n \geq 1$.

Lemma 2.4 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following statement holds:*

$$E|X_n|^\alpha I\big(|X_n| \leq b\big) \leq C\big\{E|X|^\alpha I\big(|X| \leq b\big) + b^\alpha P\big(|X| > b\big)\big\}, \tag{2.1}$$

$$E|X_n|^\alpha I\big(|X_n| > b\big) \leq CE|X|^\alpha I\big(|X| > b\big), \tag{2.2}$$

where C is a positive constant.

Lemma 2.5 (Hu [16]) *Let $b_1, b_2, \dots$ be a nondecreasing unbounded sequence of positive numbers. Let $\alpha_1, \alpha_2, \dots$ be nonnegative numbers, and $\Lambda_k = \alpha_1 + \cdots + \alpha_k$ for $k \geq 1$. Let r be a fixed positive number. Assume that for each $n \geq 1$,*

$$E\Big(\max_{1\leq k\leq n}|S_k|\Big)^r \leq C\sum_{k=1}^{n}\alpha_k. \tag{2.3}$$

If

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^r}-\frac{1}{b_{l+1}^r}\right) < \infty, \tag{2.4}$$

$$\frac{\Lambda_n}{b_n^r} \quad \text{is bounded}, \tag{2.5}$$

then

$$\lim_{n\to\infty}\frac{S_n}{b_n} = 0 \quad a.s., \tag{2.6}$$

and with the growth rate

$$\frac{S_n}{b_n} = O\left(\frac{\beta_n}{b_n}\right) \quad a.s., \tag{2.7}$$

where

$$\beta_n = \max_{1\leq k\leq n} b_k \nu_k^{\delta/r}, \quad \forall 0<\delta<1, \nu_n = \sum_{k=n}^{\infty}\frac{\alpha_k}{b_k^r}, \lim_{n\to\infty}\frac{\beta_n}{b_n} = 0. \tag{2.8}$$

And

$$E\left(\max_{1\leq l\leq n}\left|\frac{S_l}{b_l}\right|\right)^r \leq 4C\sum_{l=1}^{n}\frac{\alpha_l}{b_l^r} < \infty, \tag{2.9}$$

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{b_l}\right|\right)^r \leq 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty. \tag{2.10}$$

If we further assume that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{\beta_l}\right|\right)^r \leq 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^r} < \infty. \tag{2.11}$$

Proof It follows from Corollary 2.1.1 of Hu [16] that (2.6)-(2.8) hold. By (2.3) and Theorem 1.1 of Fazekas and Klesov [17], we have

$$E\left(\max_{1\leq l\leq n}\left|\frac{S_l}{b_l}\right|\right)^r \leq 4C\sum_{l=1}^{n}\frac{\alpha_l}{b_l^r} \leq 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty. \tag{2.12}$$

Therefore

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{b_l}\right|\right)^r = \lim_{n\to\infty} E\left(\max_{1\leq l\leq n}\left|\frac{S_l}{b_l}\right|\right)^r \leq 4C\sum_{l=1}^{\infty}\frac{\alpha_l}{b_l^r} < \infty, \tag{2.13}$$

following from the monotone convergence theorem of Rao [18]. Equation (2.11) follows from the proof of Lemma 1.2 of Hu and Hu [19]. □

3 Main results

Theorem 3.1 *Let $\{X_n, n \geq 1\}$ be a sequence of random variables and put $G(x) = \sup_{n\geq 1} P(|X_n| > x)$ for $x > 0$. Denote $Y_n = -n^{1/p} I(X_n < -n^{1/p}) + X_n I(|X_n| \leq n^{1/p}) + n^{1/p} I(X_n > n^{\frac{1}{p}})$, $n \geq 1$, where p is a positive constant. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive numbers with $b_n \uparrow \infty$. Suppose that $\{\frac{a_n}{b_n}(Y_n - EY_n), n \geq 1\}$ satisfies the maximal moment inequality with exponent 2. Assume that the following two conditions hold:*

$$\sum_{i=1}^{n} a_i = O(b_n), \tag{3.1}$$

$$\frac{a_n}{b_n} = O\left(n^{-1/p}\right) \quad \text{for some } 1 \leq p < 2. \tag{3.2}$$

If $\int_0^\infty x^{p-1} G(x)\,dx < \infty$, then

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i (X_i - EX_i) = 0 \quad a.s. \tag{3.3}$$

Proof By Lemma 2.1(i),

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) = \sum_{n=1}^{\infty} P\left(|X_n| > n^{1/p}\right) < \infty. \tag{3.4}$$

Therefore $P(X_n \neq Y_n, \text{ i.o.}) = 0$ follows from the Borel-Cantelli lemma and (3.4). Thus (3.3) is equivalent to the following:

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i (Y_i - EX_i) = 0 \quad \text{a.s.} \tag{3.5}$$

So, in order to prove (3.3), we need only to prove

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i (Y_i - EY_i) = 0 \quad \text{a.s.} \tag{3.6}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i (EX_i - EY_i) = 0. \tag{3.7}$$

Firstly, we prove (3.6). In view of Lemma 2.1(i), (ii) and (3.2), we have

$$\begin{aligned}\sum_{n=1}^{\infty} E\left(\frac{a_n}{b_n}(Y_n - EY_n)\right)^2 &\leq \sum_{n=1}^{\infty} \frac{a_n^2}{b_n^2} EY_n^2 \\ &= \sum_{n=1}^{\infty} \frac{a_n^2}{b_n^2}\left[EX_n^2 I\left(|X_n| \leq n^{1/p}\right) + n^{2/p} EI\left(|X_n| > n^{1/p}\right)\right] \\ &\ll \sum_{n=1}^{\infty}\left[n^{-2/p} EX_n^2 I\left(|X_n| \leq n^{1/p}\right) + P\left(|X_n| > n^{1/p}\right)\right] \\ &< \infty.\end{aligned}$$

Thus it follows by Lemma 2.2 that

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n}(Y_n - EY_n) \quad \text{converges a.s.} \tag{3.8}$$

By Kronecker's lemma, we can obtain (3.6) immediately.

Secondly, we prove (3.7). By (3.1) and Lemma 2.1(iii), we can get

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i EX_i I\left(|X_i| > i^{1/p}\right) = 0. \tag{3.9}$$

By (3.2) and Lemma 2.1(i), we have

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} n^{1/p} P\left(X_n > n^{1/p}\right) \ll \sum_{n=1}^{\infty} P\left(X_n > n^{1/p}\right) < \sum_{n=1}^{\infty} P\left(|X_n| > n^{1/p}\right) < \infty$$

and

$$\sum_{n=1}^{\infty} \frac{a_n}{b_n} n^{1/p} P\left(X_n < -n^{1/p}\right) < \infty.$$

Thus it follows by Kronecker's lemma that

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i i^{1/p} P\left(X_i > i^{1/p}\right) = 0 \tag{3.10}$$

and

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i i^{1/p} P\left(X_i < -i^{1/p}\right) = 0. \tag{3.11}$$

Therefore,

$$\begin{aligned}&\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i (EX_i - EY_i) \\ &\quad = \lim_{n\to\infty}\left[\frac{1}{b_n} \sum_{i=1}^{n} a_i EX_i I\left(|X_i| > i^{1/p}\right)\right.\end{aligned}$$

$$+\frac{1}{b_n}\sum_{i=1}^{n} a_i i^{1/p} P\bigl(X_i < -i^{1/p}\bigr) - \frac{1}{b_n}\sum_{i=1}^{n} a_i i^{1/p} P\bigl(X_i > i^{1/p}\bigr)\Biggr]$$

$$= 0$$

follows from (3.9)-(3.11). Hence the result is proved. □

Theorem 3.2 *Let $\{X_n, n \geq 1\}$ be a sequence of mean zero random variables, which is stochastically dominated by a random variable X. Let $\{a_n, n \geq 1\}$ and $\{b_n, n \geq 1\}$ be sequences of positive numbers with $b_n \uparrow \infty$. Put $c_n = \frac{b_n}{a_n}$ for $n \geq 1$, $1 < r < 2$. Denote $Y_n = -c_n I(X_n < -c_n) + X_n I(|X_n| \leq c_n) + c_n I(X_n > c_n)$, $n \geq 1$, and suppose that $\{\frac{a_n}{b_n}(Y_n - EY_n), n \geq 1\}$ satisfies the maximal moment inequality with exponent 2. Assume that the following two conditions hold*:

$$E|X|^r < \infty, \tag{3.12}$$

$$N(n) = Card\{i : c_i \leq n\} \ll n^r, \quad n \geq 1. \tag{3.13}$$

Then

$$\lim_{n\to\infty} \frac{1}{b_n}\sum_{i=1}^{n} a_i X_i = 0 \quad a.s. \tag{3.14}$$

Proof Let $N(0) = 0$. By (3.13), we can see that $c_n \to \infty$ as $n \to \infty$. By (3.12) and (3.13),

$$\begin{aligned}
\sum_{n=1}^{\infty} P(X_n \neq Y_n) &= \sum_{n=1}^{\infty} P\bigl(|X_n| > c_n\bigr) \ll \sum_{n=1}^{\infty} P\bigl(|X| > c_n\bigr) \\
&= \sum_{n=1}^{\infty} \sum_{c_n \leq j < c_n+1} P\bigl(|X| > c_n\bigr) \\
&\leq \sum_{j=1}^{\infty} \sum_{j-1 < c_n \leq j} P\bigl(|X| > j-1\bigr) \\
&= \sum_{j=1}^{\infty} \bigl(N(j) - N(j-1)\bigr) \sum_{k=j}^{\infty} P\bigl(k-1 < |X| \leq k\bigr) \\
&= \sum_{k=1}^{\infty} P\bigl(k-1 < |X| \leq k\bigr) \sum_{j=1}^{k} \bigl(N(j) - N(j-1)\bigr) \\
&= \sum_{k=1}^{\infty} N(k) P\bigl(k-1 < |X| \leq k\bigr) \\
&\ll \sum_{k=1}^{\infty} k^r P\bigl(k-1 < |X| \leq k\bigr) \\
&\ll E|X|^r < \infty,
\end{aligned} \tag{3.15}$$

which implies $P(X_i \neq Y_i, \text{ i.o.}) = 0$ from the Borel-Cantelli lemma. So, in order to prove (3.14), we need only to prove

$$\lim_{n\to\infty} \frac{1}{b_n} \sum_{i=1}^{n} a_i Y_i = 0 \quad \text{a.s.} \tag{3.16}$$

By (3.12), (3.13), Lemma 2.4, and the proof of (3.15), we have

$$\begin{aligned}
&\sum_{n=1}^{\infty} E\left(\frac{a_n}{b_n}(Y_n - EY_n)\right)^2 \\
&\quad \leq \sum_{n=1}^{\infty} c_n^{-2} EY_n^2 \\
&\quad = \sum_{n=1}^{\infty} c_n^{-2}\left[EX_n^2 I\left(|X_n| \leq c_n\right) + E\left(c_n^2 I\left(|X_n| > c_n\right)\right)\right] \\
&\quad \ll \sum_{n=1}^{\infty} c_n^{-2} EX^2 I\left(|X| \leq c_n\right) + \sum_{n=1}^{\infty} P\left(|X| > c_n\right) \\
&\quad = \sum_{n=1}^{\infty} \sum_{c_n \leq j < c_n + 1} c_n^{-2} EX^2 I\left(|X| \leq c_n\right) + \sum_{n=1}^{\infty} P\left(|X| > c_n\right) \\
&\quad \leq \sum_{j=1}^{\infty} \sum_{j-1 < c_n \leq j} c_n^{-2} EX^2 I\left(|X| \leq j\right) + C \\
&\quad \ll \sum_{j=2}^{\infty} \left(N(j) - N(j-1)\right)(j-1)^{-2} \sum_{k=1}^{j} EX^2 I\left(k-1 < |X| \leq k\right) + C \\
&\quad = \sum_{j=2}^{\infty} \left(N(j) - N(j-1)\right)(j-1)^{-2} \left[EX^2 I\left(0 < |X| \leq 1\right) + \sum_{k=2}^{j} EX^2 I\left(k-1 < |X| \leq k\right)\right] + C \\
&\quad = \sum_{j=2}^{\infty} \left(N(j) - N(j-1)\right)(j-1)^{-2} EX^2 I\left(0 < |X| \leq 1\right) \\
&\qquad + \sum_{k=2}^{\infty} EX^2 I\left(k-1 < |X| \leq k\right) \sum_{j=k}^{\infty} \left(N(j) - N(j-1)\right)(j-1)^{-2} + C \\
&\quad \leq \sum_{j=2}^{\infty} N(j)\left((j-1)^{-2} - j^{-2}\right) EX^2 I\left(0 < |X| \leq 1\right) \\
&\qquad + \sum_{k=2}^{\infty} EX^2 I\left(k-1 < |X| \leq k\right) \sum_{j=k}^{\infty} N(j)\left((j-1)^{-2} - j^{-2}\right) + C \\
&\quad \ll \sum_{j=2}^{\infty} j^{r-3} + \sum_{k=2}^{\infty} EX^2 I\left(k-1 < |X| \leq k\right) \sum_{j=k}^{\infty} j^{r-3} + C \\
&\quad \ll \sum_{k=2}^{\infty} k^{r-2} E\left(|X|^r k^{2-r} I\left(k-1 < |X| \leq k\right)\right) + C \\
&\quad \ll E|X|^r + C < \infty.
\end{aligned} \tag{3.17}$$

Combining Lemma 2.2, (3.17) and Kronecker's lemma, we can get

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}a_i(Y_i - EY_i) = 0 \quad \text{a.s.} \tag{3.18}$$

To complete the proof of (3.16), it suffices to show that

$$\lim_{n\to\infty}\frac{1}{b_n}\sum_{i=1}^{n}a_i EY_i = 0. \tag{3.19}$$

By (3.12), (3.13) and $EX_n = 0$, it follows that

$$\begin{aligned}\sum_{n=1}^{\infty}\left|\frac{a_n}{b_n}EY_n\right| &\le \sum_{n=1}^{\infty}c_n^{-1}\bigl[E|X_n|I\bigl(|X_n| > c_n\bigr) + E\bigl(c_n I\bigl(|X_n| > c_n\bigr)\bigr)\bigr] \\ &= \sum_{n=1}^{\infty}c_n^{-1}E|X_n|I\bigl(|X_n| > c_n\bigr) + \sum_{n=1}^{\infty}P\bigl(|X_n| > c_n\bigr).\end{aligned} \tag{3.20}$$

Observe that

$$\begin{aligned}&\sum_{n=1}^{\infty}c_n^{-1}E|X_n|I\bigl(|X_n| > c_n\bigr) \\ &\quad\le C\sum_{n=1}^{\infty}c_n^{-1}E|X|I\bigl(|X| > c_n\bigr) \\ &\quad= C\sum_{n=1}^{\infty}\sum_{c_n\le j<c_n+1}c_n^{-1}E|X|I\bigl(|X| > c_n\bigr) \\ &\quad\le C\sum_{j=1}^{\infty}\sum_{j-1<c_n\le j}c_n^{-1}E|X|I\bigl(|X| > j-1\bigr) \\ &\quad\le C\sum_{j=2}^{\infty}\bigl(N(j)-N(j-1)\bigr)(j-1)^{-1}\sum_{n=j-1}^{\infty}E|X|I\bigl(n < |X| \le n+1\bigr) \\ &\quad= C\sum_{n=1}^{\infty}E|X|I\bigl(n < |X| \le n+1\bigr)\sum_{j=2}^{n+1}\bigl(N(j)-N(j-1)\bigr)(j-1)^{-1} \\ &\quad\le C\sum_{n=1}^{\infty}E|X|I\bigl(n < |X| \le n+1\bigr)\sum_{j=2}^{n}N(j)\bigl((j-1)^{-1}-j^{-1}\bigr) \\ &\qquad+ C\sum_{n=1}^{\infty}E|X|I\bigl(n < |X| \le n+1\bigr)\frac{N(n+1)}{n} \\ &\quad\le C\sum_{n=1}^{\infty}E|X|I\bigl(n < |X| \le n+1\bigr)\sum_{j=2}^{n}j^{r-2} + C\sum_{n=1}^{\infty}n^{r-1}E|X|I\bigl(n < |X| \le n+1\bigr) \\ &\quad\le C\sum_{n=1}^{\infty}n^{r-1}E|X|I\bigl(n < |X| \le n+1\bigr)\end{aligned}$$

$$\begin{aligned}&\le C\sum_{n=1}^{\infty}E|X|^{r}I\bigl(n<|X|\le n+1\bigr)\\&\le CE|X|^{r}<\infty.\end{aligned}\tag{3.21}$$

So, we can get

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{b_n}EY_n\right|<\infty$$

from (3.15), (3.20) and (3.21). Consequently,

$$\sum_{n=1}^{\infty}\frac{a_n}{b_n}EY_n\quad\text{converges,}\tag{3.22}$$

which implies (3.19) from Kronecker's lemma. We complete the proof of theorem. □

Theorem 3.3 *Let* $\{X_n, n\ge 1\}$ *be a sequence of mean zero random variables satisfying the maximal moment inequality with exponent* 2. *Denote* $Q_n=\max_{1\le k\le n}EX_k^2$, $n\ge 1$ *and* $Q_0=0$. *For* $1\le p<2$, *assume that*

$$\sum_{n=1}^{\infty}\frac{Q_n}{n^{2/p}}<\infty.\tag{3.23}$$

Then

$$\lim_{n\to\infty}\frac{S_n}{n^{1/p}}=0\quad a.s.,\tag{3.24}$$

and with the growth rate

$$\frac{S_n}{n^{1/p}}=O\left(\frac{\beta_n}{n^{1/p}}\right)\quad a.s.,\tag{3.25}$$

where

$$\begin{aligned}&\beta_n=\max_{1\le k\le n}k^{1/p}\nu_k^{\delta/2},\quad \forall 0<\delta<1, \nu_n=\sum_{k=n}^{\infty}\frac{\alpha_k}{k^{2/p}},\\&\alpha_k=C\bigl(kQ_k-(k-1)Q_{k-1}\bigr),\quad k\ge 1, \lim_{n\to\infty}\frac{\beta_n}{n^{1/p}}=0.\end{aligned}\tag{3.26}$$

And

$$E\left(\max_{1\le l\le n}\left|\frac{S_l}{l^{1/p}}\right|^2\right)\le 4\sum_{l=1}^{n}\frac{\alpha_l}{l^{2/p}}<\infty,\tag{3.27}$$

$$E\left(\sup_{l\ge 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right)\le 4\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}}<\infty.\tag{3.28}$$

If we further assume that $\alpha_n > 0$ for infinitely many n, then

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{\beta_l}\right|^2\right) \leq 4\sum_{l=1}^{\infty}\frac{\alpha_l}{\beta_l^2} < \infty. \tag{3.29}$$

In addition, for any $0 < r < 2$,

$$E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) \leq 1 + \frac{4r}{2-r}\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty. \tag{3.30}$$

Proof Since $\{X_n, n \geq 1\}$ is a sequence of mean zero random variables satisfying the maximal moment inequality with exponent 2, we have

$$E\left(\max_{1\leq j\leq n}\left|\sum_{k=1}^{j}X_k\right|^2\right) \leq C\sum_{k=1}^{n}EX_k^2 \leq CnQ_n = \sum_{k=1}^{n}\alpha_k. \tag{3.31}$$

And we can obtain $\alpha_k \geq 0$ for all $k \geq 1$ from its definition. Denote $b_n = n^{1/p}$ and $\Lambda_n = \sum_{k=1}^{n}\alpha_k$, $n \geq 1$. By (3.23), we can get

$$\sum_{l=1}^{\infty}\Lambda_l\left(\frac{1}{b_l^2} - \frac{1}{b_{l+1}^2}\right) = C\sum_{l=1}^{\infty}lQ_l\left(\frac{1}{l^{2/p}} - \frac{1}{(l+1)^{2/p}}\right) \leq \frac{2C}{p}\sum_{l=1}^{\infty}\frac{Q_l}{l^{2/p}} < \infty. \tag{3.32}$$

Thus (2.4) holds. It follows from Remark 2.1 in [16] that (2.4) implies (2.5). By Lemma 2.5, we can get (3.24)-(3.29) immediately. It follows from (3.28) that

$$\begin{aligned}
E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r\right) &= \int_0^{\infty}P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r > t\right)dt \\
&= \int_0^{1}P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r > t\right)dt + \int_1^{\infty}P\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^r > t\right)dt \\
&\leq 1 + E\left(\sup_{l\geq 1}\left|\frac{S_l}{l^{1/p}}\right|^2\right)\int_1^{\infty}t^{-2/r}\,dt \\
&\leq 1 + \frac{4r}{2-r}\sum_{l=1}^{\infty}\frac{\alpha_l}{l^{2/p}} < \infty.
\end{aligned}$$

The proof is completed. □

Remark 3.4 It is easy to see that a wide class of mean zero random variables satisfies the maximal moment inequality with exponent 2. Examples include independent random variables, negatively associated random variables (see Matula [20]), negatively superadditive dependent random variables (see Shen *et al.* [12]), φ-mixing random variables and AANA random variables (see Wang *et al.* [21, 22]), and $\tilde{\rho}$-mixing random variables (see Utev *et al.* [23]). So Theorems 3.1-3.3 hold for this wide class of random variables.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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