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L_p Error estimate for minimal norm SBF interpolation

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Abstract
By the method of spherical splitting, the interpolation capability of the spherical basis function (SBF) is investigated. As the main result, we deduce the error estimate for the minimal norm SBF interpolation in the metric of the pth Lebesgue integral function space on the sphere. The result shows that the interpolation capability of SBF depends not only on the smoothness of the target function, but also on the geometric distributions of the interpolation knots.

MSC: 41A36; 41A25

Keywords: spherical basis function; minimal norm; L_p approximation; error estimate

1 Introductions and main results

Over the past decades, research in the field of function approximation of scattered data points gradually drifted from the polynomials to the radial basis functions (RBFs), and later to the spherical basis functions (SBFs) in a spherical coordinate. Recently, people, such as Wang and Li [1], Freeden *et al.* [2], and Muller [3], have moved their interests further to the topics of spherical approximation [1–7]. Based on the developments, spherical harmonic analysis was established and had some considerable progress. Meanwhile, L_p error estimations for the SBF approximation were studied hereafter and stepped forward in the studies of Le Gia *et al.* [4], Hubbert and Morton [5], and Sloan and Wendland [7] after 2004. In 2007, Chen [8] established error bound for the minimal norm interpolation on a sphere. With this plentiful foundation, Lin and Cao [9] embedded firstly the smooth SBFs in a native space and specified the error bound between the best approximation and the target function via L_p metric. As a consecutive study, the paper thus aims to derive a minimal norm interpolation with the L_p measure.

For a fixed integer q ($q \geq 1$), S^q is a unit sphere in R^{q+1}, *i.e.*, $S^q = \{x = (x_1, x_2, \dots, x_{q+1}) \in R^{q+1}, x_1^2 + x_2^2 + \cdots + x_{q+1}^2 = 1\}$, and $d\omega$ represents a sufficient small elemental area on the spherical surface S^q. The total surface area of S^q can hence be expressed as

$$\omega_q = \int_{S^q} d\omega = \frac{2\pi^{\frac{q+1}{2}}}{\Gamma(\frac{q+1}{2})}.$$

In regard to $d\omega$, the $L^p(S^q)$ inner product of functions f and g is given as

$$(f, g)_{L^p(S^q)} = \int_{S^q} f(x)g(x)\,d\omega(x),$$

and the $L^p(S^q)$ norm is defined as

$$\|f\|_p := \|f\|_{L^p(S^q)} := \begin{cases} (\int_{S^q} |f(x)|^p \, d\omega)^{\frac{1}{p}}, & 1 \le p < +\infty; \\ \max_{x \in S^q} |f(x)|, & p = +\infty, \end{cases}$$

where $L^p(S^q)$ represents a function space constructed by a complex function, $f : S^q \to C$, fulfilling $\|f\|_p < +\infty$. An identical function in $L^p(S^q)$ is characterized as a function identical to the functions which have the same output values everywhere with the same inputs.

For an integer $k \ge 0$, H_k^q denotes a linear subspace in R^{q+1} constructed by all the kth order homogeneous harmonic polynomials $p(x)$ restricted by S^q, and Π_n^q denotes the function constructed by all the kth order, $k \le n$, spherical harmonic polynomials. The relationship between Π_n^q and H_k^q can primarily be given as

$$\Pi_n^q = \bigoplus_{k=0}^{n} H_k^q.$$

Obviously, the dimension of H_k^q is

$$d_k^q = \dim H_k^q = \begin{cases} \frac{2k+q-1}{k+q-1} \binom{2k+q-1}{k+q-1}, & k \ge 1; \\ 1, & k = 0. \end{cases}$$

In H_k^q, we select a set of functions $Y_{k,j}$ ($j = 1, 2, \dots, d_k^q$) to form a standard orthogonal basis $\Pi_n^q = \{Y_{k,j}, j = 1, 2, \dots, d_k^q, k = 0, 1, 2, \dots, n\}$. The subspace H_k^q also confirms $L^2(S^q) = \overline{\bigoplus_{k=0}^{\infty} H_k^q}$ and the famous additive law:

$$\sum_{j=1}^{d_k^q} Y_{k,j}(\xi) Y_{k,j}(\eta) = \frac{d_k^q}{\omega_q} P_k(q+1 : \xi \cdot \eta),$$

where $\xi \cdot \eta$ denotes the inner product of ξ and η. $P_k(q+1 : x)$ denotes the kth order Legendre polynomial satisfying $P_k(q+1 : 1) = 1$ and complies with

$$\int_{-1}^{1} P_k(q+1 : x) P_l(q+1 : x) \left(1 - x^2\right)^{\frac{q}{2}-1} dx = \frac{\omega_q}{\omega_{q-1} d_k^q} \delta_{k,l}.$$

By giving $A = \{\{A_{n,j}\} : A_{n,j} \in R, n = 0, 1, 2, \dots, j = 1, \dots, d_n^q\}$, a series $\{A_n\} \in A$, $A_n > 0$, $n = 0, 1, 2, \dots$, can also be expressed as $\{A_{n,j}\} \in A$, $A_{n,j} = A_n$, $j = 1, \dots, d_n^q$. Using the series $\{A_{n,j}\}$, a space defined as

$$H^s(S^q) = \left\{ F \in L^2(S^q), \sum_{n=0}^{\infty} \sum_{j=1}^{d_n^q} A_n^s (F, Y_{n,j})^2 < \infty \right\}$$

has the inner product $(\cdot, \cdot)_{H^s(S^q)}$:

$$(F, G)_{H^s(S^q)} = \sum_{n=0}^{\infty} \sum_{j=1}^{d_n^q} A_n^s (F, Y_{n,j})(G, Y_{n,j}),$$

and the corresponding norm is consequently defined as

$$\|F\|_{H^s(S^q)} = \big((F,F)_{H^s(S^q)}\big)^{\frac{1}{2}}.$$

As known from the definition here, $H^s(S^q)$ is a Hilbert space, *i.e.*, when $s > \frac{q}{2}$, $H^s(S^q)$ has a reproducing kernel

$$K(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{d_n^q} A_n^{-s} Y_{n,j}(\xi) Y_{n,j}(\eta)$$

if A_n satisfies $A_n \sim 1 + n^2$ (see the reference [7]).

Hereafter, we assume consistently in this study that $H^s(S^q)$ is a reproducing kernel Hilbert space (RKHS) constructed by the reproducing kernel and call the space a native space of $K(\xi,\eta)$ (see [10, 11]). From the additive law, we have

$$K(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{d_n^q} A_n^{-s} Y_{n,j}(\xi) Y_{n,j}(\eta) = \sum_{n=0}^{\infty} \frac{d_n^q}{\omega_q} A_n^{-s} P_n(q+1, \xi\cdot\eta), \quad \xi,\eta \in S^q,$$

where $P_n(q+1,x)$ is the nth order Legendre polynomial. Here $K(\xi,\eta)$ is obviously a spherical radial basis function.

Let us assume that $X = \{x_1, x_2, \dots, x_N\}$ is a set of N points taken from the unit sphere S^q and has N output values $y_1, y_2, \dots, y_N$ corresponding to $x_1, x_2, \dots, x_N$ through a function F:

$$I_N(y) = \big\{F \in H^s\big(S^q\big) : F(x_i) = y_i, i = 1, 2, \dots, N\big\}.$$

If there exists $F \in I_N(y)$ such that $\|F\|$ can be minimized, we say that the minimization of F is a problem subject to a minimal norm. By denoting $S_N(\xi)$ as an interpolation with the minimal norm, there is a kernel basis expression for the interpolation $S_N(\xi)$ (see the references [2, 7, 8]):

$$S_N(\xi) = \sum_{i=0}^{N} \alpha_i K(x_i, \xi), \quad \xi \in S^q.$$

Suppose that y_i, $i = 1, 2, \dots, N$, are produced by the function $f(x)$, *i.e.*, $f(x_i) = y_i$, $i = 1, \dots, N$. The interpolation problem becomes intrinsically a function approximation problem to estimate the error between $f(x)$ and $S_N(x)$. The approximation order of the interpolation can then be determined by the grid norm, h, and the input $x_i \in X$. Here, h is defined as

$$h := \sup_{x\in S^q} \min_{x_i \in X} \operatorname{dist}(x, x_i),$$

where $\operatorname{dist}(\xi,\eta)$ is the spherical distance between ξ and η, *i.e.*, $\operatorname{dist}(\xi,\eta) = \arccos(\xi,\eta)$, $\xi,\eta \in S^q$.

Because $H^s(S^q)$ is a RKHS constituted by the reproducing kernel $K(\xi,\eta)$, the error between $f(x)$ and $S_N(x)$ in the knot x_i will vanish, *i.e.*,

$$\bigl(S_N - f, K(\cdot,x_i)\bigr)_{H^s(S^q)} = S_N(x_i) - f(x_i) = 0$$

for an arbitrary $f \in H^s(S^q)$, if S_N is interpolated via the minimal norm.

Suppose $V_x := \operatorname{span}\{K(\cdot,x_1),\dots,K(\cdot,x_N)\}$, we thus have

$$(S_N - f, v)_{H^s(S^q)} = 0$$

for a given arbitrary $v \in V_x$. It implies that S_N is an orthogonal projection of f to the span V_x in the space $H^s(S^q)$.

Furthermore, the L^p ($p = +\infty$) approximation of an interpolation via minimal norm is consequently bounded as follows. By assuming there exists a positive integer m such that the grid norm $h \le \frac{1}{2m}$ for $X = \{x_1,x_2,\dots,x_N\}$ is taken from S^q, we deduce that the L^p ($p = +\infty$) approximation via the minimal norm must satisfy

$$\sup_{x\in S^q}\bigl|f(x) - S_N(x)\bigr| \le C\left(\sum_{n=m+1}^{\infty}\frac{d_n^q}{A_n^2}\right)\|f\| \tag{1.1}$$

for an arbitrary $f \in H^s(S^q)$ (refer to [8]). Coming with these contributions, this study is sought to extend the significant results to a more general case of L^p ($1 \le p \le +\infty$).

Theorem 1.1 *Assume that $X = \{x_1,x_2,\dots,x_N\}$ is a set of N points taken from the unit sphere S^q, h is the corresponding grid norm, and there exists a positive integer m such that $h \le \frac{1}{2m}$. For an arbitrary function $f \in H^s(S^q)$, $s > \frac{q}{2}$, and its corresponding interpolation via the minimal norm S_N, there must exist a positive constant C, which is independent of f and h, such that the following estimations are satisfied*:

$$\begin{aligned}\|f - S_N\|_p &\le Ch^{s+\frac{q}{p}-\frac{q}{2}}\left(\sum_{n=m+1}^{\infty}\frac{d_n^q}{A_n^2}\right)^{\frac{1}{2}}\|f - S_N\|_{H^s(S^q)}\\ &\le Ch^{s+\frac{q}{p}-\frac{q}{2}}\left(\sum_{n=m+1}^{\infty}\frac{d_n^q}{A_n^2}\right)^{\frac{1}{2}}\|f\|_{H^s(S^q)},\quad 2 \le p < \infty\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}\|f - S_N\|_p &\le Ch^{s}\left(\sum_{n=m+1}^{\infty}\frac{d_n^q}{A_n^2}\right)^{\frac{1}{2}}\|f - S_N\|_{H^s(S^q)}\\ &\le Ch^{s}\left(\sum_{n=m+1}^{\infty}\frac{d_n^q}{A_n^2}\right)^{\frac{1}{2}}\|f\|_{H^s(S^q)},\quad 1 \le p < 2.\end{aligned} \tag{1.3}$$

Theorem 1.2 *Assume that $X = \{x_1,x_2,\dots,x_N\}$ is a set of N points taken from the unit sphere S^q, and the grid norm satisfies the condition $h \le \frac{1}{2m}$. For an arbitrary function $f \in H^s(S^q)$,*

$s > \frac{q}{2}$, and its corresponding interpolation via the minimal norm S_N, we have

$$\|f - S_N\|_p \le Ch^{2s+\frac{q}{p}-\frac{q}{2}} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{1}{2}} \|f\|_{H^{2s}(S^q)}, \quad 2 \le p < \infty \tag{1.4}$$

and

$$\|f - S_N\|_p \le Ch^{2s} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{1}{2}} \|f\|_{H^{2s}(S^q)}, \quad 1 \le p < 2, \tag{1.5}$$

where C denotes a positive constant independent of f and h.

To prove the main results of Theorems 1.1 and 1.2, the concept of spherical cap should be introduced first: With center x_0 and radius γ, the spherical cap $G(x_0, \gamma)$ is defined as

$$G(x_0, \gamma) = \{x \in S^q, x \cdot x_0 > \cos\gamma\},$$

and the surface area, referred to as $G(\gamma)$ of $G(x, \gamma)$, can be given as

$$G(\gamma) = \int_0^{\gamma} \omega_q \sin^{q-1}\theta \, d\theta. \tag{1.6}$$

2 Lemmas

To completely prove Theorems 1.1 and 1.2, five lemmas are given as follows.

Lemma 2.1 (refer to [5, 6, 9]) *Assume an integer $q \ge 1$, constants $M = 2\sqrt{q}$ and $\delta_q = \frac{1}{(4q+1)^{\frac{3}{2}}}$, and $h_0 := \frac{\theta}{M+M_1+\delta_q}$ with an arbitrary positive number M_1, and $\theta \in (0, \frac{\pi}{3})$, then there exists a point set $Z_h \subset S^q$ with an arbitrary $h \in (0, h_0)$ satisfying*

$$S^q = \bigcup_{z \in Z_h} G(z, Mh). \tag{2.1}$$

If F_A represents the characteristic function of a given set $A \in S^q$, there exists a positive integer Q independent of h satisfying

$$\sum_{z \in Z_h} F_{G(z, \bar{M}h)} \le Q, \tag{2.2}$$

where $\bar{M} = M + M_1$. Furthermore, there exists a constant C_Q independent of h such that $|Z_h| \le C_Q h^{-q}$.

Lemma 2.2 (refer to [5, 6, 9]) *By giving constants $S \ge 0$, M_1, C, $\bar{M} = M + M_1$, $h_0 = \frac{C}{3\bar{M}}$ and $h \in (0, h_0)$, there must exist an arbitrary $f \in H^s(S^q)$ such that*

$$\sum_{z \in Z_h} \|f\|^2_{H^s(G(z, \bar{M}h))} \le Q \|f\|^2_{H^s(S^q)}. \tag{2.3}$$

Lemma 2.3 *Assume that η_1^* is an evaluation functional corresponding to η_1 in $H^s(S^q)$, η_1^* in $H^s(S^q)$ can alternatively be expressed as $K(\eta_1, \xi)$.*

Lemma 2.4 *Assume that* $x^*, x_1^*, x_2^*, \dots, x_N^*$ *are evaluation functionals in* $H^s(S^q)$ *and their expressions in* $H^s(S^q)$ *are* $u, u_1, \dots, u_N$. *Together with* S_N, *which is interpolated via the minimal norm along* $x_1, x_2, \dots, x_N$, *and* z, *which is the optimized approximation of* u *in the* $\operatorname{span}\{u_1, u_2, \dots, u_N\}$, *we have*

$$\bigl|f(x) - S_N(x)\bigr| \le \|u - z\| \|f - s_N\|_2. \tag{2.4}$$

It should be noted that Lemmas 2.3 *and* 2.4 *can be directly obtained from* [9].

Lemma 2.5 (refer to [4, 5, 7]) *There exist two constants* $C > 0$ *and* $h_1 > 0$, *dependent only on* s *and* d, *such that a function* g ($g \in H^s(S^q)$, $g|_X = 0$ *for arbitrary* $X \in S^d$, $h \le h_1$) *satisfies*

$$\|g\|_2 \le Ch^s \|g\|_{H^s(S^q)}, \tag{2.5}$$

when $s > \frac{d}{2}$.

3 Proof of theorems

Proof of Theorem 1.1 From Lemma 2.1, we have

$$\begin{aligned}\|f - S_N\|_p^p &= \int_{S^q} \bigl|(f - S_N)(\xi)\bigr|^p \, d\omega(\xi) \\ &\le \sum_{z \in Z_h} \int_{G(z,Mh)} \bigl|(f - S_N)(\xi)\bigr|^p \, d\omega(\xi)\end{aligned}$$

for arbitrary $1 \le p < \infty$ and $M = 2\sqrt{q}$. We consider first the local error estimation. Since $f - S_N$ is continuous on $\overline{G(z,Mh)}$, and $\overline{G(z,Mh)}$ is a compact support of S^q, there must exist $\xi_z \in \overline{G(z,Mh)}$ such that $f - S_N$ is maximized at ξ_z. We consequently have

$$\begin{aligned}\|f - S_N\|_p^p &\le \sum_{z \in Z_h} \bigl|(f - S_N)(\xi_z)\bigr|^p \int_{G(z,Mh)} d\omega(\xi) \\ &\le C_q h^q \sum_{z \in Z_h} \bigl|(f - S_N)(\xi_z)\bigr|^p,\end{aligned}$$

where C_q is a constant dependent only on q and satisfying $G(Mh) = C_q h^q$.

(a) When $p \ge 2$: Taking ξ_z^* as an evaluation functional, $S_{\xi_z}^*$ is its best approximation in $\operatorname{span}\{u_1, u_2, \dots, u_N\}$. From the Jensen inequality (refer to [9, 12]), $\sum_{i=1}^N a_i^p \le (\sum_{i=1}^N a_i^2)^{\frac{p}{2}}$, Lemmas 2.3 and 2.4, we obtain

$$\begin{aligned}\|f - S_N\|_p^p &\le Ch^q \Bigl(\sum_{z \in Z_h} \bigl|(f - S_N)(\xi_z)\bigr|^2 \Bigr)^{\frac{p}{2}} \\ &\le Ch^q \Bigl(\sum_{z \in Z_h} \bigl\|\xi_z^* - S_{\xi_z}^*\bigr\|^2 \|f - S_N\|_{L^2(G(z,Mh))}^2 \Bigr)^{\frac{p}{2}} \\ &\le Ch^q \Biggl(\sum_{z \in Z_h} \|f - S_N\|_{L^2(G(z,Mh))}^2 \Bigl\| K(\xi_z, \cdot) - \sum_{i=1}^N \alpha_i K(x_i, \cdot) \Bigr\|^2 \Biggr)^{\frac{p}{2}}\end{aligned}$$

$$\leq Ch^q \left(\sum_{z \in Z_h} \|f - S_N\|^2_{L^2(G(z,Mh))} \right)^{\frac{p}{2}} \left(\sum_{z \in Z_h} \left\| K(\xi_z, \cdot) - \sum_{i=1}^{N} \alpha_i K(x_i, \cdot) \right\|^2 \right)^{\frac{p}{2}}$$

$$\leq Ch^q \left(\sum_{z \in Z_h} \|f - S_N\|^2_{L^2(G(z,Mh))} \right)^{\frac{p}{2}} \left(\sum_{z \in Z_h} \sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{p}{2}}.$$

It should be noted that one of the following inequalities (refer to [8]) is used in the last step of the derivations above

$$\left\| K(x, \cdot) - \sum_{i=1}^{N} \alpha_i K(x_i, \cdot) \right\|^2 \leq C \sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2}. \tag{3.1}$$

For sure, by taking $\bar{h_0} = \frac{1}{2m}$ (where $\bar{h_0} = \min\{h_0, h_1\}$), we can go further

$$\|f - S_N\|_p^p \leq Ch^{q+sp} \left(\sum_{z \in Z_h} \|f - S_N\|^2_{H^s(G(z,Mh))} \right)^{\frac{p}{2}} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{p}{2}} C_Q h^{-\frac{pq}{2}}$$

$$\leq Ch^{q+sp-\frac{pq}{2}} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{p}{2}} \|f - S_N\|^p_{H^s(S^q)}$$

with $h \in (0, \bar{h_0})$. By using both $M < \bar{M}$ and the consequence $|Z_h| \leq C_Q h^{-q}$, Lemma 2.5, Lemma 2.2, and Lemma 2.1, we achieve

$$\|f - S_N\|_p \leq Ch^{s+\frac{q}{p}-\frac{q}{2}} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{1}{2}} \|f - S_N\|_{H^s(S^q)}. \tag{3.2}$$

(b) When $1 \leq p < 2$: From the inequality (refer to [9, 12])

$$\sum_{i=1}^{N} a_i^p \leq N^{1-\frac{p}{2}} \left(\sum_{i=1}^{N} a_i^2 \right)^{\frac{p}{2}},$$

and the fact which Z_h is less than Ch^{-q}, we obtain first

$$\|f - S_N\|_p^p \leq Ch^q \left(\sum_{z \in Z_h} \left| (f - S_N)(\xi_z) \right|^2 \right)^{\frac{p}{2}} h^{-(1-\frac{p}{2})q}$$

$$\leq Ch^{\frac{pq}{2}} \left(\sum_{z \in Z_h} \left\| \xi_z^* - S_{\xi_z}^* \right\|^2 \|f - S_N\|^2_{L^2(G(z,Mh))} \right)^{\frac{p}{2}}$$

$$\leq Ch^{sp} \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{p}{2}} \|f - S_N\|^p_{H^s(S^q)}.$$

Following similarly the steps in the case of $p \geq 2$, we also have

$$\|f - S_N\|_p \leq Ch^s \left(\sum_{n=m+1}^{\infty} \frac{d_n^q}{A_n^2} \right)^{\frac{1}{2}} \|f - S_N\|_{H^s(S^q)}. \tag{3.3}$$

With the fact that S_N is the orthogonal projection of f in $H^s(S^q)$, the proof of Theorem 1.1 is thus completed. □

Proof of Theorem 1.2 From the property of orthogonal projection of S_N, the Cauchy-Schwarz inequality, and Lemma 2.5, we can easily obtain

$$\begin{aligned}
\|f-S_N\|^p_{H^s(S^q)} &= \left(\|f-S_N\|^2_{H^s(S^q)}\right)^{\frac{p}{2}} \\
&= (f-S_N, f-S_N)^{\frac{p}{2}}_{H^s(S^q)} \\
&= (f-S_N, f)^{\frac{p}{2}}_{H^s(S^q)} \\
&= \left(\sum_{n=0}^{\infty}\sum_{j=1}^{d_n^q} A_n^s (f-S_N, Y_{n,j})(f, Y_{n,j})\right)^{\frac{p}{2}} \\
&\leq \left(\left(\sum_{n=0}^{\infty}\sum_{j=1}^{d_n^q} A_n^{2s}(f, Y_{n,j})^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty}\sum_{j=1}^{d_n^q}(f-S_N, Y_{n,j})^2\right)^{\frac{1}{2}}\right)^{\frac{p}{2}} \\
&= \left(\|f\|_{H^{2s}(S^q)}\|f-S_N\|_{L_2(S^q)}\right)^{\frac{p}{2}} \\
&\leq Ch^{\frac{sp}{2}}\|f\|^{\frac{p}{2}}_{H^{2s}(S^q)}\|f-S_N\|^{\frac{p}{2}}_{H^s(S^q)}.
\end{aligned}$$

Therefore

$$\|f-S_N\|^p_{H^s(S^q)} \leq Ch^{sp}\|f\|^p_{H^{2s}(S^q)}. \tag{3.4}$$

Together with the results of Theorem 1.1, Theorem 1.2 is thus established. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
In this paper, JW had the main role in providing derivations. CY and ZG also contributed significantly with their corresponding effort to finish the work.

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References
1. Wang, KY, Li, LQ: Harmonic Analysis and Approximation on the Unit Sphere. Science Press, Beijing (2000)
2. Freeden, W, Gervens, T, Schreiner, M: Constructive Approximation on the Sphere. Oxford University Press, New York (1998)
3. Müller, C: Spherical Harmonics. Lecture Notes in Mathematics, vol. 17. Springer, New York (1966)
4. Le Gia, QL, Narcowich, F, Ward, J, Wendland, H: Continuous and discrete least-squares approximation by basis functions on spheres. J. Approx. Theory **143**, 124-133 (2006)
5. Hubbert, S, Morton, TM: L_p-Error estimates for radial basis function interpolation on the sphere. J. Approx. Theory **129**, 58-77 (2004)
6. Hubbert, S, Morton, TM: A Duchon framework for the sphere. J. Approx. Theory **129**, 28-57 (2004)
7. Sloan, LH, Wendland, H: Inf-sup condition for spherical polynomials and radial basis functions on spheres. Math. Comput. **267**, 1319-1331 (2009)

8. Chen, ZX: Error estimates of minimal norm interpolant approximation by the spherical radial basis functions. Acta Math. Appl. Sin. **30**(1), 159-167 (2007) (in Chinese)
9. Lin, SB, Cao, FL: L^p Approximation by shifts of smooth kernels on spheres. Chin. Ann. Math., Ser. A **31**(5), 615-624 (2010) (in Chinese)
10. Wendland, H: Scattered Data Approximation. Cambridge University Press, Cambridge (2005)
11. Narcowich, F, Schaback, R, Ward, J: Approximation in Sobolev spaces by kernel expansions. J. Approx. Theory **114**(1), 70-83 (2002)
12. Brenner, SC, Scott, RL: The Mathematical Theory of Finite Element Methods. Springer, New York (1994)

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