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Fixed point theorems for generalized contractions on *GP*-metric spaces

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Abstract

In this paper, we present two fixed point theorems on mappings, defined on *GP*-complete *GP*-metric spaces, which satisfy a generalized contraction property determined by certain upper semi-continuous functions. Furthermore, we illustrate applications of our theorems with a number of examples. Inspired by the work of Jachymski, we also establish equivalences of certain auxiliary maps in the context of *GP*-complete *GP*-metric spaces. **MSC:** 47H10; 54H25

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1 Introduction and preliminaries

In 1922, Stefan Banach [1] stated his celebrated theorem on the existence and uniqueness of a fixed point of certain self-maps defined on certain metric spaces for the first time. Specifically, this elegant theorem, also known as the Banach contraction mapping principle, can be formulated as follows: any mapping $T : (X,d) \rightarrow (X,d)$ has a unique point $x \in X$ such that Tx = x provided that there exists a constant $k \in (0,1)$ satisfying the inequality $d(Tx, Ty) \leq kd(x, y)$ for every $x, y \in X$, where (X, d) is a complete metric space. A mapping T for which the inequality mentioned above holds is called a contraction.

Since its first appearance, the Banach contraction mapping principle has become the main tool to study contractions as they appear abundantly in a wide array of quantitative sciences. Its most well-known application is in ordinary differential equations, particularly, in the proof of the Picard-Lindelöf theorem which guarantees the existence and uniqueness of solutions of first-order initial value problems. It is worth emphasizing that the remarkable strength of the Banach principle originates from the constructive process it provides to identify the fixed point. This notable strength further attracted the attention of not only many prominent mathematicians studying in many branches of mathematics related to nonlinear analysis, but also many researchers who are interested in iterative methods to examine the quantitative problems involving certain mappings and space structures required in their work in various areas such as social sciences, biology, economics, and computer sciences.

Indeed, in 1994, Matthews, a computer scientist who is an expert on semantics, announced in [2] an analog of Banach's principle in a new space he called a partial metric space. Matthews's innovative approach was quickly adopted and improved by fixed point theorists (see, *e.g.*, [3–27]) with the aim of discovering analogs of Banach's principle in



© 2013 Bilgili et al.; licensee Springer. This is an Open Access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/2.0), which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited. the context of partial metric spaces to broaden its applications and enrich the fixed point theory as a result.

A closer look to the work of these distinguished mathematicians after Matthews's studies reveals that their discoveries can be categorized in terms of the techniques implemented to produce the analogs of Banach's principle. The first technique is to introduce new space structures with certain properties which guarantee the existence and/or uniqueness of fixed points of contractions. In addition to Matthews's investigations, cone metric spaces, *D*-metric spaces, and *G*-metric spaces (see, *e.g.*, [28–44]) constitute a few of the examples to the first approach. The second technique is to introduce mappings defined on metric spaces satisfying certain new contractive conditions. For example, cyclic contractions and weak ϕ contractions can be listed as a few.

As another example to the first approach mentioned above, Zand and Nezhad [43] recently introduced *GP*-metric spaces which are a combination of the notions of partial metric spaces and *G*-metric spaces. Then they proved a number of fixed point theorems on these new spaces for certain type of contractions. In this paper, we exercise the second approach by using the space structure they initiated to prove certain fixed point theorems for generalized contractions. First, we review the necessary notation, definitions, and fundamental results produced on *GP*-metric spaces that we will need in this work.

Definition 1.1 [43] Let *X* be a non-empty set. A function $G_p : X \times X \times X \longrightarrow [0, +\infty)$ is called a *GP*-metric if the following conditions are satisfied:

- (GP1) x = y = z if $G_p(x, y, z) = G_p(z, z, z) = G_p(y, y, y) = G_p(x, x, x)$;
- (GP2) $0 \le G_p(x, x, x) \le G_p(x, x, y) \le G_p(x, y, z)$ for all $x, y, z \in X$;
- (GP3) $G_p(x, y, z) = G_p(x, z, y) = G_p(y, z, x) = \cdots$, symmetry in all three variables;
- (GP4) $G_p(x, y, z) \le G_p(x, a, a) + G_p(a, y, z) G_p(a, a, a)$ for any $x, y, z, a \in X$.

Then the pair (X, G) is called a *GP*-metric space.

Example 1.1 [43] Let $X = [0, \infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Then (X, G_p) is a *GP*-metric space.

Proposition 1.1 [43] *Let* (X, G_p) *be a GP-metric space, then for any x, y, z and a* \in *X, it follows that*

- (i) $G_p(x, y, z) \le G_p(x, x, y) + G_p(x, x, z) G_p(x, x, x);$
- (ii) $G_p(x, y, y) \le 2G_p(x, x, y) G_p(x, x, x);$
- (iii) $G_p(x, y, z) \le G_p(x, a, a) + G_p(y, a, a) + G_p(z, a, a) 2G_p(a, a, a);$
- (iv) $G_p(x, y, z) \le G_p(x, a, z) + G_p(a, y, z) G_p(a, a, a)$.

Proposition 1.2 [43] *Every GP-metric space* (X, G_p) *defines a metric space* (X, D_{G_p}) *where*

 $D_{G_p}(x, y) = G_p(x, y, y) + G_p(y, x, x) - G_p(x, x, x) - G_p(y, y, y) \quad for \ all \ x, y \in X.$

Definition 1.2 [43] Let (X, G_p) be a *GP*-metric space and let $\{x_n\}$ be a sequence of points of *X*. A point $x \in X$ is said to be the limit of the sequence $\{x_n\}$ or $x_n \to x$ if

 $\lim_{n,m\to\infty}G_p(x,x_m,x_n)=G_p(x,x,x).$

Proposition 1.3 [43] Let (X, G_p) be a GP-metric space. Then, for any sequence $\{x_n\}$ in X and a point $x \in X$, the following are equivalent:

- (A) $\{x_n\}$ is GP-convergent to x;
- (B) $G_p(x_n, x_n, x) \to G_p(x, x, x) \text{ as } n \to \infty;$
- (C) $G_p(x_n, x, x) \to G_p(x, x, x) \text{ as } n \to \infty.$

Definition 1.3 [43] Let (X, G_p) be a *GP*-metric space.

- (S1) A sequence $\{x_n\}$ is called a *GP*-Cauchy if and only if $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m)$ exists (and is finite);
- (S2) A *GP*-partial metric space (X, G_p) is said to be *GP*-complete if and only if every *GP*-Cauchy sequence in *X* is *GP*-convergent to $x \in X$ such that $G_p(x, x, x) = \lim_{m,n\to\infty} G_p(x_n, x_m, x_m).$

Now, we introduce the following.

Definition 1.4 Let (X, G_p) be a *GP*-metric space.

- (M1) A sequence $\{x_n\}$ is called 0-*GP*-Cauchy if and only if $\lim_{m,n\to\infty} G_p(x_n, x_m, x_m) = 0$;
- (M2) A *GP*-metric space (X, G_p) is said to be 0-*GP*-complete if and only if every 0-*GP*-Cauchy sequence in *X GP*-converges to a point $x \in X$ such that $G_p(x, x, x) = 0$.

Example 1.2 Let $X = [0, +\infty)$ and define $G_p(x, y, z) = \max\{x, y, z\}$ for all $x, y, z \in X$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Moreover, if $X = \mathbb{Q} \cap [0, +\infty)$ (where \mathbb{Q} denotes a set of rational numbers), then (X, G_p) is a 0-*GP*-complete *GP*-metric space.

Lemma 1.1 (See [45]) Let (X, G_p) be a GP-metric space. Then

- (A) If $G_p(x, y, z) = 0$, then x = y = z;
- (B) If $x \neq y$, then $G_p(x, y, y) > 0$.

In the rest of this paper, we will denote the positive natural numbers by \mathbb{N}^* and the natural numbers by \mathbb{N} .

2 Main results

In this section, we present our findings on fixed point theorems on 0-*GP*-complete *GP*metric spaces. We first start with the following definition.

Definition 2.1 Let (X, G_p) be a *GP*-metric space and $T : (X, G_p) \to (X, G_p)$ be a map. Let M(x, y, y) denote the value

$$\max\left\{G_{p}(x, y, y), G_{p}(x, Tx, Tx), G_{p}(y, Ty, Ty), \frac{1}{2}\left[G_{p}(x, Ty, Ty) + G_{p}(y, Tx, Tx)\right]\right\}$$
(1)

for all $x, y \in X$.

Lemma 2.1 If (X, G_p) is a GP-metric space and $T : X \to X$ is a map, then, for each $x \in X$, we have

$$M(x, Tx, Tx) = \max\{G_p(x, Tx, Tx), G_p(Tx, T^2x, T^2x)\}.$$
(2)

Proof Let $x \in X$. Then

$$\max\{G_{p}(x, Tx, Tx), G_{p}(Tx, T^{2}x, T^{2}x)\}$$

$$\leq M(x, Tx, Tx)$$

$$= \max\{G_{p}(x, Tx, Tx), G_{p}(x, Tx, Tx), G_{p}(Tx, T^{2}x, T^{2}x), \frac{1}{2}[G_{p}(x, T^{2}x, T^{2}x) + G_{p}(Tx, Tx, Tx)]\}$$

$$\leq \max\{G_{p}(x, Tx, Tx), G_{p}(x, Tx, Tx), G_{p}(Tx, T^{2}x, T^{2}x), \frac{1}{2}[G_{p}(x, Tx, Tx) + G_{p}(Tx, T^{2}x, T^{2}x)]\}$$

$$= \max\{G_{p}(x, Tx, Tx), G_{p}(Tx, T^{2}x, T^{2}x)\}.$$
(3)

The proof is complete.

Lemma 2.2 Let (X, G_p) be a GP-metric space and let $T: X \to X$ be a map such that

$$G_p(Tx, Ty, Ty) \le \phi(M(x, y, y))$$
(4)

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is a function such that $\phi(t) < t$ for all t > 0. If $x \in X$ satisfies $T^n x \neq T^{n+1} x$ for all $n \in \mathbb{N}$, then the following hold:

- (a) $M(T^n x, T^{n+1} x, T^{n+1} x) = G_p(T^n x, T^{n+1} x, T^{n+1} x)$ for all $n \in \mathbb{N}$;
- (b) $G_p(T^nx, T^{n+1}x, T^{n+1}x) \le \phi(G_p(T^{n-1}x, T^nx, T^nx)) < G_p(T^{n-1}x, T^nx, T^nx)$ for all *n* ∈ N*.

Proof (a) From Lemma 2.1, we have

$$M(T^{n}x, T^{n+1}x, T^{n+1}x) = \max\{G_{p}(T^{n}x, T^{n+1}x, T^{n+1}x), G_{p}(T^{n+1}x, T^{n+2}x, T^{n+2}x)\}.$$

Since $T^n x \neq T^{n+1}x$ for all $n \in \mathbb{N}$, then by Lemma 1.1(B), we get $G_p(T^n x, T^{n+1}x, T^{n+1}x) > 0$ for all $n \in \mathbb{N}$. Consequently, $M(T^n x, T^{n+1}x, T^{n+1}x) > 0$. Now by condition (4), we deduce that

$$G_{p}(T^{n+1}x, T^{n+2}x, T^{n+2}x) \leq \phi(M(T^{n}x, T^{n+1}x, T^{n+1}x))$$

$$< M(T^{n}x, T^{n+1}x, T^{n+1}x), \qquad (5)$$

that is, $G_p(T^{n+1}x, T^{n+2}x, T^{n+2}x) < G_p(T^nx, T^{n+1}x, T^{n+1}x)$. Hence, (a) holds.

Clearly, (b) follows from (4), (a), and the fact that $G_p(T^{n-1}x, T^nx, T^nx) > 0$ for all $n \in \mathbb{N}$.

Definition 2.2 A function $\phi : [0, \infty) \to [0, \infty)$ is called upper semi-continuous from the right if for each $t \ge 0$ and each sequence $(t_n)_{n \in \mathbb{N}^*}$ such that $t_n \ge t$ and $\lim_{n \to \infty} t_n = t$, the equality holds $\limsup_{n \to \infty} \phi(t_n) \le \phi(t)$.

Theorem 2.1 Let (X, G_p) be a GP-complete GP-metric space and let $T : X \to X$ be a map such that

$$G_p(Tx, Ty, Ty) \le \phi \big(M(x, y, y) \big), \tag{6}$$

for all $x, y \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous function from the right such that $\phi(t) < t$ for all t > 0. Then T has a unique fixed point $z \in X$. Moreover, $G_p(z, z, z) = 0$.

Proof Let $x \in X$. If there is $n \in \mathbb{N}$ such that $T^n x = T^{n+1}x$, then $T^n x$ is a fixed point of T and the uniqueness of $T^n x$ follows as in the last part of the proof below. Hence, we assume that $T^n x \neq T^{n+1}x$ for all $n \in \mathbb{N}$. Put $x_0 = x$ and construct the sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Thus, $x_{n+1} = Tx_n$ and $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. Define the sequence $\{s_n\}$ by $s_n = G_p(x_n, x_{n+1}, x_{n+1})$ for all $n \in \mathbb{N}$. From Lemma 2.2(b) we know that $\{s_n\}$ is a non-increasing sequence. Hence, there exists $c \in \mathbb{R}^+$ such that $s_n \to c$ as $n \to \infty$. We will show that c must be equal to 0. Let c > 0. By taking limitsup as $n \to \infty$ in condition (b) of Lemma 2.2, we get that

$$c = \limsup_{n \to \infty} G_p(x_n, x_{n+1}, x_{n+1}) = \limsup_{n \to \infty} \phi\left(G_p(x_n, x_{n+1}, x_{n+1})\right)$$
(7)

and so, by upper semi-continuity from the right of the function ϕ , we deduce

$$c = \limsup_{n \to \infty} \phi\left(G_p(x_n, x_{n+1}, x_{n+1})\right) \le \phi(c) < c,\tag{8}$$

which is a contradiction. Hence, c = 0. Consequently, $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$. Next we show that $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$. Assume the contrary. Then there exist $\varepsilon > 0$ and sequences $(n_k)_{k\in\mathbb{N}^*}$, $(m_k)_{k\in\mathbb{N}^*}$ in \mathbb{N}^* with $m_k \ge n_k \ge k$ and such that $G_p(x_{n_k}, x_{m_k}, x_{m_k}) \ge \varepsilon$ for all $k \in \mathbb{N}^*$. From the fact that $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$, we can suppose, without loss of generality, that $G_p(x_{n_k}, x_{m_{k-1}}, x_{m_{k-1}}) < \varepsilon$. For each $k \in \mathbb{N}^*$, we have

$$\varepsilon \le G_p(x_{n_k}, x_{m_k}, x_{m_k}) \le G_p(x_{n_k}, x_{m_{k-1}}, x_{m_{k-1}}) + G_p(x_{m_{k-1}}, x_{m_k}, x_{m_k})$$

$$< \varepsilon + G_p(x_{m_{k-1}}, x_{m_k}, x_{m_k}),$$
(9)

and hence $\lim_{k\to\infty} G_p(x_{n_k}, x_{m_k}, x_{m_k}) = \varepsilon$. Now, let $k_0 \in \mathbb{N}^*$ be such that $G_p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) < \varepsilon$ ε and $G_p(x_{m_k}, x_{m_k+1}, x_{m_k+1}) < \varepsilon$ for all $k \ge k_0$. Then

$$\begin{split} G_p(x_{n_k}, x_{m_k}, x_{m_k}) &\leq M(x_{n_k}, x_{m_k}, x_{m_k}) \\ &= \max \left\{ G_p(x_{n_k}, x_{m_k}, x_{m_k}), G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}), \\ G_p(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), \\ &\frac{1}{2} \Big[G_p(x_{n_k}, x_{m_{k+1}}, x_{m_{k+1}}) + G_p(x_{m_k}, x_{n_{k+1}}, x_{n_{k+1}}) \Big] \right\} \\ &\leq \max \left\{ G_p(x_{n_k}, x_{m_k}, x_{m_k}), G_p(x_{n_k}, x_{n_{k+1}}, x_{n_{k+1}}), \\ G_p(x_{m_k}, x_{m_{k+1}}, x_{m_{k+1}}), \right. \end{split}$$

$$\frac{1}{2} \Big[G_p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + G_p(x_{n_k+1}, x_{n_k+2}, x_{n_k+2}) + \cdots \\
+ G_p(x_{m_k}, x_{m_k+1}, x_{m_k+1}) + G_p(x_{m_k}, x_{m_k-1}, x_{m_k-1}) \\
+ G_p(x_{m_k-1}, x_{m_k-2}, x_{m_k-2}) + \cdots + G_p(x_{n_k+2}, x_{n_k+1}, x_{n_k+1}) \Big] \Big\}$$
(10)

for all $k \ge k_0$. So, $\lim_{k\to\infty} M(x_{n_k}, x_{m_k}, x_{m_k}) = \varepsilon$. Since $M(x_{n_k}, x_{m_k}, x_{m_k}) \ge \varepsilon$ for all $k \in \mathbb{N}^*$ and ϕ is upper semi-continuous from the right, we deduce that $\limsup_{k\to\infty} \phi(M(x_{n_k}, x_{m_k}, x_{m_k})) \le \phi(\varepsilon)$. On the other hand, for each $k \in \mathbb{N}^*$, we have

$$\varepsilon \leq G_p(x_{n_k}, x_{m_k}, x_{m_k}) \leq G_p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + G_p(x_{n_k+1}, x_{m_k+1}, x_{m_k+1}) + G_p(x_{m_k+1}, x_{m_k}, x_{m_k}) \leq G_p(x_{n_k}, x_{n_k+1}, x_{n_k+1}) + \phi \left(M(x_{n_k}, x_{m_k}, x_{m_k}) \right) + G_p(x_{m_k+1}, x_{m_k}, x_{m_k}),$$
(11)

so $\varepsilon \leq \limsup_{k\to\infty} \phi(M(x_{n_k}, x_{m_k}, x_{m_k})) \leq \phi(\varepsilon)$, a contradiction because $\phi(\varepsilon) < \varepsilon$. Consequently, $\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = 0$ and thus $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the *GP*-complete *GP*-metric space (X, G_p) . Hence, there is $z \in X$ such that

$$\lim_{n,m\to\infty} G_p(x_n, x_m, x_m) = \lim_{n\to\infty} G_p(z, x_n, x_n) = G_p(z, z, z) = 0.$$
 (12)

We show that *z* is a fixed point of *T*. To this end, we first note that $G_p(z, Tz, Tz) = \lim_{n\to\infty} M(z, x_n, x_n)$, so $\limsup_{n\to\infty} \phi(M(z, x_n, x_n)) \le \phi(G_p(z, Tz, Tz))$. On the other hand, since for each $n \in \mathbb{N}$, $G_p(z, Tz, Tz) \le G_p(z, x_n, x_n) + G_p(x_n, Tz, Tz)$, it follows that

$$G_{p}(z, Tz, Tz) \leq \limsup_{n \to \infty} (G_{p}(z, x_{n}, x_{n}) + G_{p}(x_{n}, Tz, Tz))$$

$$= \limsup_{n \to \infty} G_{p}(x_{n}, Tz, Tz)$$

$$\leq \limsup_{n \to \infty} \phi(M(x_{n-1}, z, z))$$

$$\leq \phi(G_{p}(z, Tz, Tz)).$$
(13)

Therefore, $G_p(z, Tz, Tz) = 0$ and thus z = Tz. Finally, let $u \in X$ be such that Tu = u. Then

$$G_p(u,z,z) = G_p(Tu,Tz,Tz) \le \phi(M(u,z,z)) = \phi(G_p(u,z,z)).$$

$$(14)$$

Hence, $G_p(u, z, z) = 0$, *i.e.*, u = z. This concludes the proof.

Definition 2.3 Let $\psi(\cdot, \cdot) : X \times X \to [0, \infty)$ be a given function. Then Q(x, y, z) will denote the value

$$\max\left\{G_{p}(x, y, z), G_{p}(x, Tx, Tx), G_{p}(y, Ty, Ty), G_{p}(z, Tz, Tz), \frac{1}{2}[G_{p}(x, Ty, Ty) + G_{p}(y, Tx, Tx)], \frac{1}{2}[G_{p}(x, Tz, Tz) + G_{p}(z, Tx, Tx)]\right\}.$$
(15)

Then we obtain the following statement.

Corollary 2.1 Let (X, G_p) be a GP-complete GP-metric space and let $T : X \to X$ be a map such that

$$0 \leq \left[\psi(z,z) - \psi(y,y)\right] \max\left\{G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\right\}$$
$$+ \psi(x,y)\psi(x,z) \left[\phi\left(Q(x,y,z)\right) - G_p(Tx,Ty,Tz)\right]$$

for all $x, y, z \in X$, where $\phi : [0, \infty) \to [0, \infty)$ is an upper semi-continuous from the right function such that $\phi(t) < t$ for all t > 0. Then T has a unique fixed point $z \in X$. Moreover, $G_p(z, z, z) = 0$.

Proof Clearly, by taking y = z in the hypothesis, we have

$$G_p(Tx, Ty, Ty) \le \phi(Q(x, y, y)) = \phi(M(x, y, y))$$

for all $x, y \in X$. Then the conditions of Theorem 2.1 hold. This concludes the proof. \Box

Example 2.1 Let $X = [0, \infty)$, $G_p : X \times X \times X \to \mathbb{R}$ be defined by $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Let $f : X \to X$ be defined by $Tx = \frac{x}{2(x+1)}$ and $\psi(t) = \frac{t}{1+t}$ for all $t \in [0, +\infty)$.

Proof Without loss of generality, we assume that $x \ge y$. Then

$$G_p(Tx, Ty, Ty) = \frac{x}{2(x+1)} \leq \frac{x}{x+1} = \psi(x) = \psi\left(G_p(x, y, y)\right).$$

Then the condition of Theorem 2.1 holds and *T* has a unique fixed point 0 in $[0, \infty)$. Moreover, $G_p(0, 0, 0) = 0$.

Example 2.2 Let $X = [0, \infty)$, $G_p : X \times X \times X \to \mathbb{R}$ be defined by $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \le x < 1/3, \\ (1-x)/2 & \text{if } 1/3 \le x \le 1, \\ \frac{1}{4}x & \text{if } x > 1 \end{cases}$$

and $\psi(t) = \frac{1}{2}t$ for all $t \in [0, +\infty)$.

Proof To prove this example, we need to consider the following cases:

• Let $0 \le x, y < 1/3$. Then

$$G_p(Tx, Ty, Ty) = \frac{1}{2} \max\{x^2, y^2\} \le \frac{1}{2} \max\{x, y\} = \psi(G_p(x, y, y))$$

• Let $1/3 \le x, y \le 1$. Then

$$G_p(Tx, Ty, Ty) = \frac{1}{2} \max\{1 - x, 1 - y\} \le \frac{1}{2} \max\{x, y\} = \psi(G_p(x, y, y)).$$

$$G_p(Tx, Ty, Ty) = \frac{1}{4} \max\{x, y\} \le \frac{1}{2} \max\{x, y\} = \psi(G_p(x, y, y)).$$

• Let $0 \le x < 1/3$ and $1/3 \le y \le 1$. Then

$$G_p(Tx, Ty, Ty) = \max\left\{\frac{1}{2}x^2, (1-y)/2\right\} \le \frac{1}{2}\max\{x, y\} = \psi(G_p(x, y, y)).$$

• Let $0 \le x < 1/3$ and y > 1. Then

$$G_p(Tx, Ty, Ty) = \max\left\{\frac{1}{2}x^2, \frac{1}{4}y\right\} \le \frac{1}{2}\max\{x, y\} = \psi(G_p(x, y, y)).$$

• Let $1/3 \le x \le 1$ and y > 1. Then

$$G_p(Tx, Ty, Ty) = \max\left\{(1-x)/2, \frac{1}{4}y\right\} \le \frac{1}{2}\max\{x, y\} = \psi\left(G_p(x, y, y)\right).$$

Then the condition of Theorem 2.1 holds and *T* has a unique fixed point 0 in $[0, \infty)$. Moreover, $G_p(0, 0, 0) = 0$.

Lemma 2.3 Let $\phi : [0, \infty) \to [0, \infty)$ be nondecreasing and let t > 0. If $\lim_{n\to\infty} \phi^n(t) = 0$, then $\phi(t) < t$.

Theorem 2.2 Let (X, G_p) be a GP-complete GP-metric space and $T : X \to X$ be a map such that

$$G_p(Tx, Ty, Ty) \le \phi(N(x, y, y)), \tag{16}$$

where $N(x, y, y) = \max\{G_p(x, y, y), G_p(x, Tx, Tx), G_p(y, Ty, Ty)\}$ for all $x, y \in X$, and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all t > 0. Then T has a unique fixed point $z \in X$. Moreover, $G_p(z, z, z) = 0$.

Proof Let $x \in X$. If there is $n \in \mathbb{N}$ such that $T^n x = T^{n+1}x$, then $T^n x$ is a fixed point of T and the uniqueness of $T^n x$ follows as in the last part of the proof below. Hence, we will assume that $T^n x \neq T^{n+1}x$ for all $n \in \mathbb{N}$. Put $x_0 = x$ and construct the sequence $(x_n)_{n \in \mathbb{N}}$, where $x_n = T^n x_0$ for all $n \in \mathbb{N}$. Thus, $x_{n+1} = Tx_n$ and $G_p(x_n, x_{n+1}, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. By Lemma 2.2(b),

$$G_p(x_n, x_{n+1}, x_{n+1}) \le \phi \left(G_p(x_{n-1}, x_n, x_n) \right) \tag{17}$$

for all $n \in \mathbb{N}$. Then, since ϕ is nondecreasing, we deduce that

$$G_p(x_n, x_{n+1}, x_{n+1}) \le \phi^n \big(G_p(x_0, x_1, x_1) \big) \tag{18}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} G_p(x_n, x_{n+1}, x_{n+1}) = 0$. Now, choose an arbitrary $\varepsilon > 0$. Since $\lim_{n\to\infty} \phi^n(\varepsilon) = 0$, it follows from Lemma 2.3 that $\phi(\varepsilon) < \varepsilon$, so there is $n_{\varepsilon} \in \mathbb{N}^*$ such that

$$G_p(x_n, x_{n+1}, x_{n+1}) < \varepsilon - \phi(\varepsilon) \tag{19}$$

for all $n \ge n_{\varepsilon}$. Therefore,

$$G_{p}(x_{n}, x_{n+2}, x_{n+2}) \leq G_{p}(x_{n}, x_{n+1}, x_{n+1}) + G_{p}(x_{n+1}, x_{n+2}, x_{n+2})$$

$$< \varepsilon - \phi(\varepsilon) + \phi(G_{p}(x_{n}, x_{n+1}, x_{n+1}))$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon$$
(20)

for all $n \ge n_{\varepsilon}$. So,

$$G_{p}(x_{n}, x_{n+3}, x_{n+3}) \leq G_{p}(x_{n}, x_{n+1}, x_{n+1}) + G_{p}(x_{n+1}, x_{n+3}, x_{n+3})$$

$$< \varepsilon - \phi(\varepsilon) + \phi(N(x_{n}, x_{n+2}, x_{n+2}))$$

$$\leq \varepsilon - \phi(\varepsilon) + \phi(\varepsilon) = \varepsilon, \qquad (21)$$

and following this process,

$$G_p(x_n, x_{n+k}, x_{n+k}) < \varepsilon \tag{22}$$

for all $n \ge n_{\varepsilon}$ and $k \in \mathbb{N}^*$. Consequently,

$$\lim_{n,m\to\infty}G_p(x_n,x_m,x_m)=0,$$
(23)

and thus $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in the G_p -complete G_p -metric space (X, G_p) . Hence, there is $z \in X$ such that

$$0 = \lim_{n,m \to \infty} G_p(x_n, x_m, x_m) = \lim_{n \to \infty} G_p(z, x_n, x_n) = G_p(z, z, z).$$
(24)

We show that *z* is a fixed point of *T*. Assume the contrary. Then $G_p(z, Tz, Tz) > 0$. For each $n \in \mathbb{N}^*$, we have

$$G_p(z, Tz, Tz) \le G_p(z, x_n, x_n) + G_p(x_n, Tz, Tz) \le G_p(z, x_n, x_n) + \phi(N(z, z, x_{n-1})).$$
(25)

From our assumption that $G_p(z, Tz, Tz) > 0$, it easily follows that there is $n_0 \in \mathbb{N}$ such that $N(z, z, x_{n-1}) = G_p(z, Tz, Tz)$ for all $n \ge n_0$. So,

$$G_p(z, Tz, Tz) \le G_p(z, x_n, x_n) + \phi \left(G_p(z, Tz, Tz) \right)$$
(26)

for all $n \ge n_0$. Taking limits as $n \to \infty$, we obtain that

$$G_p(z, Tz, Tz) \le \phi \left(G_p(z, Tz, Tz) \right) < G_p(z, Tz, Tz),$$
(27)

a contradiction. Consequently, z = Tz. Finally, the uniqueness of z follows as in Theorem 2.1.

Example 2.3 Let X = [0,1], $G_p : X \times X \times X \to \mathbb{R}$ be defined by $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Let $T : X \to X$ be defined by

$$Tx = \left(x^2 - x^4\right)/8$$

and $\psi(t) = \frac{1}{4}t^2$ for all $t \in [0, +\infty)$.

Proof Without loss of generality, we assume that $x \ge y$. Then

$$N(x, y, y) = \max \{G_p(x, y, y), G_p(x, Tx, Tx), G_p(y, Ty, Ty)\} = x.$$

And so

$$G_p(Tx, Ty, Ty) = \max\left\{ (x^2 - x^4)/8, (y^2 - y^4)/8 \right\}$$

= $\max\left\{ \frac{(x - x^2)(x + x^2)}{8}, \frac{(y - y^2)(y + y^2)}{8} \right\}$
 $\leq \max\left\{ \frac{x(x - x^2)}{4}, \frac{y(y - y^2)}{4} \right\}$
 $\leq \frac{1}{4} \max\{x^2, y^2\} = \frac{1}{4}x^2 = \psi(N(x, y, y)).$

Then the condition of Theorem 2.2 holds and *T* has a unique fixed point 0 in [0,1]. Moreover, $G_p(0,0,0) = 0$.

Example 2.4 Let $X = [0, \infty)$, $G_p : X \times X \times X \to \mathbb{R}$ be defined by $G_p(x, y, z) = \max\{x, y, z\}$. Then (X, G_p) is a *GP*-complete *GP*-metric space. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} \frac{1}{4}(x^2 + x) & \text{if } 0 \le x < 1, \\ \frac{1}{8}x & \text{if } x \ge 1 \end{cases}$$

and $\psi(t) = \frac{1}{2}t$ for all $t \in [0, +\infty)$.

Proof To prove this example, we need to examine the following cases:

• Let $0 \le x, y < 1$. Then $N(x, y, y) = \max\{x, y\}$ and

$$G_p(Tx, Ty, Ty) = \max\left\{\frac{1}{4}(x^2 + x), \frac{1}{4}(y^2 + y)\right\} \le \frac{1}{2}\max\{x, y\} = \psi(N(x, y, y)).$$

• Let $x, y \ge 1$. Then $N(x, y, y) = \max\{x, y\}$ and

$$G_p(Tx, Ty, Ty) = \frac{1}{8} \max\{x, y\} \le \frac{1}{2} \max\{x, y\} = \psi(N(x, y, y)).$$

• Let $0 \le x < 1$ and $y \ge 1$. Then N(x, y, y) = y and

$$G_p(Tx, Ty, Ty) = \max\left\{\frac{1}{4}(x^2 + x), \frac{1}{8}y\right\} \le \frac{1}{2}\max\{x, y\} = \frac{1}{2}y = \psi(N(x, y, y)).$$

Then the condition of Theorem 2.2 holds and *T* has a unique fixed point 0 in [0,1]. Moreover, $G_p(0,0,0) = 0$.

Corollary 2.2 Let (X, G_p) be a GP-complete GP-metric space and $T : X \to X$ be a map such that

$$G_p(Tx, Ty, Tz) \le \phi(N(x, y, z)), \tag{28}$$

where $N(x, y, y) = \max\{G_p(x, y, z), G_p(x, Tx, Tx), G_p(y, Ty, Ty), G_p(z, Tz, Tz)\}$ for all $x, y \in X$, and $\phi : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all t > 0. Then T has a unique fixed point $z \in X$. Moreover, $G_p(z, z, z) = 0$.

Similarly, we have the corollary below.

Corollary 2.3 Let (X, G_p) be a GP-complete GP-metric space and $T : X \to X$ be a map such that

$$0 \leq \left[\psi(z,z) - \psi(y,y)\right] \max\left\{G_p(x,x,x), G_p(y,y,y), G_p(z,z,z)\right\}$$
$$+ \psi(x,y)\psi(x,z) \left[\phi\left(R(x,y,z)\right) - G_p(Tx,Ty,Tz)\right],$$

where $R(x, y, z) = \max\{G_p(x, y, z), G_p(x, Tx, Tx), G_p(y, Ty, Ty), G_p(z, Tz, Tz)\}$ for all $x, y, z \in X$, and $\phi : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function such that $\lim_{n\to\infty} \phi^n(t) = 0$ for all t > 0. Then T has a unique fixed point $z \in X$. Moreover, $G_p(z, z, z) = 0$.

In [46], Jachymski proved the equivalence of auxiliary functions (see Lemma 1). Inspired by the results from this remarkable paper of Jachymski, we finish this paper by stating the following theorem.

Theorem 2.3 (See [46]) Let (X, G_p) be a GP-complete GP-metric space and $T : X \to X$ be a self-mapping. Assume that

$$\begin{split} M(x,y,y) &= \max \left\{ G_p(x,y,y), G_p(x,Tx,Tx), G_p(y,Ty,Ty), \right. \\ &\left. \frac{1}{2} \big[G_p(x,Ty,Ty) + G_p(y,Tx,Tx) \big] \right\}. \end{split}$$

Then the following statements are equivalent:

(i) there exist functions $\psi, \eta \in \Psi$ such that

$$\psi(G(Tx, Ty, Ty)) \leq \psi(M(x, y, y)) - \eta(M(x, y, y))$$

for any $x, y \in X$;

(ii) there exists a function β: [0,∞) → [0,1] such that for any bounded sequence {t_n} of positive reals, β(t_n) → 1 implies t_n → 0 and

$$G(Tx, Ty, Ty) \le \beta (M(x, y, y)) \psi (M(x, y, y))$$

for any $x, y \in X$;

(iii) there exists a continuous function $\eta : [0, \infty) \to [0, \infty)$ such that $\eta^{-1}(\{0\}) = 0$ and

$$G(Tx, Ty, Ty) \le M(x, y, y) - \eta \big(M(x, y, y) \big)$$

for any $x, y \in X$;

(iv) there exist a function $\psi \in \Psi$ and a nondecreasing, right-continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(t) < t$ and for all t > 0 with

$$\psi(G(Tx, Ty, Ty)) \leq \varphi(\psi(M(x, y, y)))$$

for any $x, y \in X$;

(v) there exists a continuous and nondecreasing function $\varphi : [0, \infty) \to [0, \infty)$ such that $\varphi(t) < t$ and for all t > 0 with

$$\psi(G(Tx, Ty, Ty)) \le \varphi(M(x, y, y))$$

for any $x, y \in X$.

Competing interests

Authors' contributions

The authors declare that they have no competing interests.

All authors read and approved the final manuscript.

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