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Approximation of solutions to an equilibrium problem in a nonuniformly smooth Banach space

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Abstract

An equilibrium problem based on a projection algorithm is investigated. A strong convergence theorem for solutions of the equilibrium problems is established in a nonuniformly smooth Banach space.

Keywords: equilibrium problem; fixed point; quasi-ϕ-nonexpansive mapping; projection

1 Introduction

Recently, equilibrium problems have been studied as an effective and powerful tool for studying a wide class of real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and networks; see [1–16] and the references therein. It is well known that equilibrium problems include many important problems in nonlinear analysis and optimization such as the Nash equilibrium problem, variational inequalities, complementarity problems, vector optimization problems, fixed point problems, saddle point problems, and game theory. For the solutions of equilibrium problems, there are several algorithms to solve the problem; see [17–28] and the references therein. However, most of these results are obtained in the framework of Hilbert spaces or uniformly smooth Banach spaces.

The purpose of this paper is to study solution problems of an equilibrium problem based on a projection algorithm in a nonuniformly smooth Banach space. The organization of this paper is as follows. In Section 2, we provide some necessary preliminaries. In Section 3, a projection algorithm is introduced and the convergence analysis is given. A strong convergence theorem is established in a nonuniformly smooth Banach space. Applications of the main results are also discussed in this section.

2 Preliminaries

Let E be a real Banach space, and let E^* be the dual space of E. We denote by J the normalized duality mapping from E to 2^{E^*} defined by

$$Jx = \{f^* \in E^* : \langle x, f^* \rangle = \|x\|^2 = \|f^*\|^2\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is said to be uniformly convex if $\lim_{n\to\infty}\|x_n - y_n\| = 0$ for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that $\|x_n\| = \|y_n\| = 1$ and $\lim_{n\to\infty}\|\frac{x_n+y_n}{2}\| = 1$. Let $U_E = \{x \in E : \|x\| = 1\}$ be the unit sphere of E. Then the Banach space E is said to be smooth provided

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in U_E$. It is also said to be uniformly smooth if the above limit is attained uniformly for $x, y \in U_E$. It is well known that if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E. It is also well known that E is uniformly smooth if and only if E^* is uniformly convex.

Recall that a Banach space E enjoys the Kadec-Klee property if for any sequence $\{x_n\} \subset E$ and $x \in E$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then $\|x_n - x\| \to 0$ as $n \to \infty$. For more details on the Kadec-Klee property, the readers can refer to [29] and the references therein. It is well known that if E is a uniformly convex Banach space, then E enjoys the Kadec-Klee property. In this paper, we use $\to$ and $\rightharpoonup$ to denote the strong convergence and weak convergence, respectively.

Let C be a nonempty subset of E. Recall that a mapping $Q : C \to E^*$ is said to be monotone iff

$$\langle x - y, Qx - Qy\rangle \ge 0, \quad \forall x, y \in C.$$

$Q : C \to E^*$ is said to be α-inverse-strongly monotone iff there exists a positive real number α such that

$$\langle x - y, Qx - Qy\rangle \ge \alpha\|Qx - Qy\|^2, \quad \forall x, y \in C.$$

Recall also that a monotone mapping Q is said to be maximal iff its graph $\mathrm{Graph}(Q) = \{(x,f) : f \in Qx\}$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping Q is maximal iff for $(x,f) \in E \times E^*$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in \mathrm{Graph}(Q)$ implies $f \in Qx$. An operator Q from C into E is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0, 1]$ into E defined by $f(t) = Q(tx + (1 - t)y)$ is continuous with respect to the weak* topology of E^*. Let f be a bifunction from $C \times C$ to $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. In this paper, we investigate the following equilibrium problem. Find $p \in C$ such that

$$f(p, y) \ge 0, \quad \forall y \in C. \tag{2.1}$$

We use $EP(f)$ to denote the solution set of equilibrium problem (2.1). That is,

$$EP(f) = \{p \in C : f(p, y) \ge 0, \forall y \in C\}.$$

Given a mapping $Q : C \to E^*$, let

$$f(x, y) = \langle Qx, y - x\rangle, \quad \forall x, y \in C.$$

Then $p \in EP(f)$ iff p is a solution of the following variational inequality. Find p such that

$$\langle Qp, y - p\rangle \geq 0, \quad \forall y \in C. \tag{2.2}$$

In order to study the solution problem of equilibrium problem (2.1), we assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$, $\forall x \in C$;

(A2) f is monotone, *i.e.*, $f(x,y) + f(y,x) \leq 0$, $\forall x, y \in C$;

(A3)

$$\limsup_{t\downarrow 0} f\big(tz + (1-t)x, y\big) \leq f(x,y), \quad \forall x, y, z \in C;$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and weakly lower semi-continuous.

As we all know, if C is a nonempty closed convex subset of a Hilbert space H and $P_C : H \to C$ is the metric projection of H onto C, then P_C is nonexpansive. This fact actually characterizes Hilbert spaces and consequently it is not available in more general Banach spaces. In this connection, Alber [30] recently introduced a generalized projection operator Π_C in a Banach space E which is an analogue of the metric projection P_C in Hilbert spaces.

Next, we assume that E is a smooth Banach space. Consider the functional defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

Observe that in a Hilbert space H, the equality is reduced to $\phi(x,y) = \|x-y\|^2$, $x, y \in H$. The generalized projection $\Pi_C : E \to C$ is a map that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(x,y)$, that is, $\Pi_C x = \bar{x}$, where $\bar{x}$ is the solution to the minimization problem

$$\phi(\bar{x}, x) = \min_{y\in C} \phi(y, x).$$

The existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(x,y)$ and strict monotonicity of the mapping J; see, for example, [29] and [30]. In Hilbert spaces, $\Pi_C = P_C$. It is obvious from the definition of a function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \leq \phi(x,y) \leq \big(\|y\| + \|x\|\big)^2, \quad \forall x, y \in E, \tag{2.3}$$

and

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \quad \forall x, y, z \in E. \tag{2.4}$$

Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T. A point p in C is said to be an asymptotic fixed point of T iff C contains a sequence $\{x_n\}$ which converges weakly to p such that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. The set of asymptotic fixed points of T will be denoted by $\widetilde{F}(T)$. T is said to be relatively nonexpansive iff $\widetilde{F}(T) = F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$. T is said to be quasi-ϕ-nonexpansive iff $F(T) \neq \emptyset$ and $\phi(p, Tx) \leq \phi(p, x)$ for all $x \in C$ and $p \in F(T)$.

Remark 2.1 The class of quasi-ϕ-nonexpansive mappings is more general than the class of relatively nonexpansive mappings which requires the restriction: $F(T) = \widetilde{F}(T)$.

Remark 2.2 The class of quasi-ϕ-nonexpansive mappings is a generalization of quasi-nonexpansive mappings in Hilbert spaces.

In this paper, we investigate the solution problem of equilibrium problem (2.1) based on a projection algorithm. A strong convergence theorem for solutions of the equilibrium problems is established in a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property.

In order to give our main results, we need the following lemmas.

Lemma 2.3 [30] *Let C be a nonempty, closed, and convex subset of a smooth Banach space E, and $x \in E$. Then $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.4 [30] *Let E be a reflexive, strictly convex, and smooth Banach space, let C be a nonempty, closed, and convex subset of E, and $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \leq \phi(y, x), \quad \forall y \in C.$$

Lemma 2.5 [30] *Let E be a reflexive, strictly convex, and smooth Banach space. Then*

$$\phi(x, y) = 0 \quad \Longleftrightarrow \quad x = y.$$

The following lemma can be obtained from [15] and [31].

Lemma 2.6 *Let C be a closed convex subset of a smooth, strictly convex, and reflexive Banach space E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4). *Let $r > 0$ and $x \in E$. Then*

(a) *There exists $z \in C$ such that*

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \quad \forall y \in C.$$

(b) *Define a mapping $T_r : E \to C$ by*

$$S_r x = \left\{ z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle, \forall y \in C \right\}.$$

Then the following conclusions hold:

(1) $F(S_r) = EP(f)$;

(2) *S_r is a firmly nonexpansive-type mapping, i.e., for all $x, y \in E$,*

$$\langle S_r x - S_r y, JS_r x - JS_r y \rangle \leq \langle S_r x - S_r y, Jx - Jy \rangle;$$

(3) *S_r is single-valued*;

(4)

$$\phi(q, S_r x) + \phi(S_r x, x) \leq \phi(q, x), \quad \forall q \in F(S_r);$$

(5) *EP(f) is closed and convex*;
(6) *S_r is quasi-ϕ-nonexpansive.*

3 Main results

Theorem 3.1 *Let E be a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property. Let C be a nonempty, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *such that EF(f) is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \text{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n \in C \text{ such that } f(y_n, y) + \frac{1}{r_n}\langle y - y_n, Jy_n - Jx_n\rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. *Then the sequence $\{x_n\}$ converges strongly to $\Pi_{EP(f)} x_1$, where $\Pi_{EP(f)}$ is the generalized projection from E onto EP(f).*

Proof First, we show that C_n is closed and convex, that the projection on it is well defined. It is obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for some $m \in \mathbb{N}$. We next prove that C_{m+1} is also closed and convex for the same m. Let For $z_1, z_2 \in C_{m+1}$, we see that $z_1, z_2 \in C_m$. It follows that $z = tz_1 + (1-t)z_2 \in C_m$, where $t \in (0,1)$. Notice that

$$\phi(z_1, y_m) \leq \phi(z_1, x_m) \quad \text{and} \quad \phi(z_1, y_h) \leq \phi(z_1, x_m).$$

The above inequalities are equivalent to

$$2\langle z_1, Jx_m - Jy_m\rangle \leq \|x_m\|^2 - \|y_m\|^2 \tag{3.1}$$

and

$$2\langle z_2, Jx_m - Jy_m\rangle \leq \|x_m\|^2 - \|y_m\|^2. \tag{3.2}$$

Multiplying t and $(1-t)$ on both sides of (3.1) and (3.2), respectively, yields that

$$2\langle z, Jx_m - Jy_m\rangle \leq \|x_m\|^2 - \|y_m\|^2.$$

That is,

$$\phi(z, y_m) \leq \phi(z, x_m).$$

This gives that C_{m+1} is closed and convex. Then C_n is closed and convex. Now, we are in a position to prove that $EP(f) \subset C_n$. $EP(f) \subset C_1 = C$ is obvious. Suppose that $EP(f) \subset C_m$ for some $m \in \mathbb{N}$. Fix $p \in EP(f) \subset C_m$. It follows that

$$\phi(p, y_m) = \phi(p, S_{r_m} x_m) \leq \phi(p, x_m),$$

which implies that $p \in C_{m+1}$. This proves that $EP(f) \subset C_n$. In the light of $x_n = \Pi_{C_n} x_1$, from Lemma 2.3, we find that $\langle x_n - z, Jx_1 - Jx_n \rangle \geq 0$ for any $z \in C_n$. It follows from $EP(f) \subset C_n$ that

$$\langle x_n - w, Jx_1 - Jx_n \rangle \geq 0, \quad \forall w \in EP(f). \tag{3.3}$$

It follows from Lemma 2.4 that

$$\begin{aligned} \phi(x_n, x_1) &= \phi(\Pi_{C_n} x_1, x_1) \\ &\leq \phi(\Pi_{F(T)} x_1, x_1) - \phi(\Pi_{F(T)} x_1, x_n) \\ &\leq \phi(\Pi_{F(T)} x_1, x_1). \end{aligned}$$

This implies that the sequence $\{\phi(x_n, x_1)\}$ is bounded. It follows from (2.3) that the sequence $\{x_n\}$ is also bounded. Since the space is reflexive, we may assume that $x_n \rightharpoonup \bar{x}$. Since C_n is closed and convex, we find that $\bar{x} \in C_n$. On the other hand, we see from the weak lower semicontinuity of the norm that

$$\begin{aligned} \phi(\bar{x}, x_1) &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jx_1 \rangle + \|x_1\|^2 \\ &\leq \liminf_{n\to\infty} \bigl(\|x_n\|^2 - 2\langle x_n, Jx_1 \rangle + \|x_1\|^2\bigr) \\ &= \liminf_{n\to\infty} \phi(x_n, x_1) \\ &\leq \limsup_{n\to\infty} \phi(x_n, x_1) \\ &\leq \phi(\bar{x}, x_1), \end{aligned}$$

which implies that $\lim_{n\to\infty} \phi(x_n, x_1) = \phi(\bar{x}, x_1)$. Hence, we have $\lim_{n\to\infty} \|x_n\| = \|\bar{x}\|$. In view of the Kadec-Klee property of E, we find that $x_n \to \bar{x}$ as $n \to \infty$. Next, we show that $p \in EF(f)$. In view of construction of $x_{n+1} = \Pi_{EP(f)} x_1 \in C_{n+1} \subset C_n$, we find that

$$\begin{aligned} \phi(x_{n+1}, x_n) &= \phi(x_{n+1}, \Pi_{C_n} x_1) \\ &\leq \phi(x_{n+1}, x_1) - \phi(\Pi_{C_n} x_1, x_1) \\ &= \phi(x_{n+1}, x_1) - \phi(x_n, x_1). \end{aligned} \tag{3.4}$$

Since $x_n = \Pi_{C_n} x_1$ and $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1} \subset C_n$, we arrive at $\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1)$. This shows that $\{\phi(x_n, x_1)\}$ is nondecreasing. It follows from the boundedness that $\lim_{n\to\infty} \phi(x, x_1)$ exists. This implies from (3.4) that

$$\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0. \tag{3.5}$$

Since $x_{n+1} = \Pi_{C_{n+1}} x_1 \in C_{n+1}$, we arrive at $\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$. It follows that

$$\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0.$$

In view of (2.3), we see that $\lim_{n\to\infty}(\|x_{n+1}\| - \|y_n\|) = 0$. This in turn implies that $\lim_{n\to\infty} \|y_n\| = \|\bar{x}\|$. It follows that $\lim_{n\to\infty} \|Jy_n\| = \|J\bar{x}\|$. This implies that $\{Jy_n\}$ is bounded. Note that both E and E^* are reflexive. We may assume that $Jy_n \rightharpoonup y^* \in E^*$. In view of the reflexivity of E, we see that $J(E) = E^*$. This shows that there exists an element $y \in E$ such that $Jy = y^*$. It follows that

$$\begin{aligned} \phi(x_{n+1}, y_n) &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|y_n\|^2 \\ &= \|x_{n+1}\|^2 - 2\langle x_{n+1}, Jy_n\rangle + \|Jy_n\|^2. \end{aligned}$$

Taking $\liminf_{n\to\infty}$ on both sides of the equality above yields that

$$\begin{aligned} 0 &\geq \|\bar{x}\|^2 - 2\langle \bar{x}, y^*\rangle + \|y^*\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|Jy\|^2 \\ &= \|\bar{x}\|^2 - 2\langle \bar{x}, Jy\rangle + \|y\|^2 \\ &= \phi(\bar{x}, y). \end{aligned}$$

That is, $\bar{x} = y$, which in turn implies that $y^* = J\bar{x}$. It follows that $Jy_n \rightharpoonup J\bar{x} \in E^*$. Since E^* enjoys the Kadec-Klee property, we obtain that $\lim_{n\to\infty} Jy_n = J\bar{x}$. Notice that

$$\|Jx_n - Jy_n\| \leq \|Jx_n - J\bar{x}\| + \|J\bar{x} - Jy_n\|.$$

It follows that $\lim_{n\to\infty} \|Jx_n - Jy_n\| = 0$. From the restriction on the sequence $\{r_n\}$, we find that

$$\lim_{n\to\infty} \frac{\|Jx_n - Jy_n\|}{r_n} = 0. \tag{3.6}$$

In view of $y_n = S_{r_n} x_n$, we see that

$$f(y_n, y) + \frac{1}{r_n}\langle y - y_n, Jy_n - Jx_n\rangle \geq 0, \quad \forall y \in C.$$

It follows from (A2) that

$$\|y - y_n\| \frac{\|Jy_n - Jx_n\|}{r_n} \geq \frac{1}{r_n}\langle y - y_n, Jy_n - Jx_n\rangle \geq -f(y_n, y) \geq f(y, y_n), \quad \forall y \in C.$$

By taking the limit as $n \to \infty$ in the above inequality, from (A4) we obtain that

$$f(y, \bar{x}) \leq 0, \quad \forall y \in C.$$

For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\bar{x}$. It follows that $y_t \in C$, which yields that $f(y_t, \bar{x}) \leq 0$. It follows from (A1) and (A4) that

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \bar{x}) \leq tf(y_t, y).$$

That is,

$$f(y_t, y) \geq 0.$$

Letting $t \downarrow 0$, we obtain from (A3) that $f(\bar{x}, y) \geq 0$, $\forall y \in C$. This implies that $\bar{x} \in EP(f)$.

Finally, we prove that $\bar{x} = \Pi_{EP(f)} x_1$. Letting $n \to \infty$ in (3.3), we see that

$$\langle \bar{x} - w, Jx_1 - J\bar{x} \rangle \geq 0, \quad \forall w \in EP(f).$$

In the light of Lemma 2.3, we find that $\bar{x} = \Pi_{EP(f)} x_1$. This completes the proof. □

We remark that L^p, where $p > 1$ is a space which satisfies the restriction in Theorem 3.1. Since every uniformly convex and uniformly smooth Banach space is a reflexive, strictly convex, and smooth Banach space such that both E and E^* have the Kadec-Klee property, we find from Theorem 3.1 the following result.

Corollary 3.2 *Let E be a uniformly convex and uniformly smooth Banach space. Let C be a nonempty, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *such that $EF(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = \Pi_{C_1} x_0, \\ y_n \in C \textit{ such that } f(y_n, y) + \frac{1}{r_n} \langle y - y_n, Jy_n - Jx_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. *Then the sequence $\{x_n\}$ converges strongly to $\Pi_{EP(f)} x_1$, where $\Pi_{EP(f)}$ is the generalized projection from E onto $EP(f)$.*

In the framework of Hilbert spaces, we find from Theorem 3.1 the following result.

Corollary 3.3 *Let E be a Hilbert space. Let C be a nonempty, closed, and convex subset of E. Let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying* (A1)-(A4) *such that $EF(f)$ is nonempty. Let $\{x_n\}$ be a sequence generated in the following manner*:

$$\begin{cases} x_0 \in E \quad \textit{chosen arbitrarily}, \\ C_1 = C, \\ x_1 = P_{C_1} x_0, \\ y_n \in C \textit{ such that } f(y_n, y) + \frac{1}{r_n} \langle y - y_n, y_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|z - y_n\|^2 \leq \|z - x_n\|^2\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \end{cases}$$

where $\{r_n\}$ is a real sequence such that $\liminf_{n\to\infty} r_n > 0$. *Then the sequence $\{x_n\}$ converges strongly to $P_{EP(f)} x_1$, where $P_{EP(f)}$ is the metric projection from E onto $EP(f)$.*

Proof Notice that $\phi(x,y) = \|x-y\|^2$. The generalized metric projection is reduced to the metric projection and the normalized duality mapping J is reduced to the identity mapping I in Hilbert spaces. The result can be obtained from Theorem 3.1 immediately. □

Competing interests
The author declares that they have no competing interests.

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