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Strong convergence results for weighted sums of $\tilde{\rho}$-mixing random variables

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Abstract
In this paper, the authors study the strong convergence for weighted sums of $\tilde{\rho}$-mixing random variables without assumption of identical distribution. As an application, the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of $\tilde{\rho}$-mixing random variables is obtained.
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1 Introduction

Many useful linear statistics based on a random sample are weighted sums of independent and identically distributed random variables. Examples include least-squares estimators, nonparametric regression function estimators and jackknife estimates, among others. In this respect, studies of strong laws for these weighted sums have demonstrated significant progress in probability theory with applications in mathematical statistics. The main purpose of the paper is to further study the strong laws for these weighted sums of $\tilde{\rho}$-mixing random variables.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Write $\mathcal{F}_S = \sigma\,(X_i, i \in S \subset \mathbb{N})$. Given σ-algebras $\mathcal{B}$, $\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup_{X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})} \frac{|EXY - EXEY|}{(\operatorname{Var} X \operatorname{Var} Y)^{1/2}}.$$

Define the $\tilde{\rho}$-mixing coefficients by

$$\tilde{\rho}(k) = \sup\bigl\{\rho(\mathcal{F}_S, \mathcal{F}_T) : \text{finite subsets } S, T \subset \mathbb{N}, \text{ such that } \operatorname{dist}(S, T) \geq k\bigr\}, \quad k \geq 0.$$

Obviously, $0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1$ and $\tilde{\rho}(0) = 1$.

Definition 1.1 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be $\tilde{\rho}$-mixing if there exists $k \in \mathbb{N}$ such that $\tilde{\rho}(k) < 1$.

The concept of the coefficient $\tilde{\rho}$ was introduced by Moore [1], and Bradley [2] was the first who introduced the concept of $\tilde{\rho}$-mixing random variables to limit theorems. Since then, many applications have been found. See, for example, Bradley [2] for the central limit theorem, Bryc and Smolenski [3], Peligrad and Gut [4], and Utev and Peligrad [5] for moment inequalities, Gan [6], Kuczmaszewska [7], Wu and Jiang [8] and Wang *et al.* [9] for almost sure convergence, Peligrad and Gut [4], Gan [6], Cai [10], Kuczmaszewska [11], Zhu [12], An and Yuan [13] and Wang *et al.* [14] for complete convergence, Peligrad [15] for invariance principle, Wu and Jiang [16] for strong limit theorems for weighted product sums of $\tilde{\rho}$-mixing sequences of random variables, Wu and Jiang [17] for Chover-type laws of the k-iterated logarithm, Wu [18] for strong consistency of estimator in linear model, Wang *et al.* [19] for complete consistency of the estimator of nonparametric regression models, Wu *et al.* [20] and Guo and Zhu [21] for complete moment convergence, and so forth. When these are compared with the corresponding results of independent random variable sequences, there still remains much to be desired. So, studying the limit behavior of $\tilde{\rho}$-mixing random variables is of interest.

Let $\{X_i, i \geq 1\}$ be a sequence of independent observations from a population distribution. A common expression for these linear statistics is $T_n \doteq \sum_{i=1}^{n} a_{ni}X_i$, where the weights a_{ni} are either real constants or random variables independent of X_i. Using an observation of the Bernstein's inequality by Cheng [22], Bai *et al.* [23] established an extension of the Hardy-Littlewood strong law for linear statistics T_n. This complements a result of Cuzick [24, Theorem 2.2]. For more details about the strong law for linear statistics T_n, one can refer to Bai and Cheng [25], Sung [26, 27], Cai [28], Jing and Liang [29], Zhou *et al.* [30], Wang *et al.* [31–33] and Wu and Chen [34], and so forth.

Recently, Sung [26] obtained the following strong convergence result for weighted sums of identically distributed negatively associated random variables.

Theorem 1.1 *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed negatively associated random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\alpha} = O(n) \tag{1.1}$$

for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha} \log^{1/\gamma} n$ for some $\gamma > 0$. Furthermore, suppose that $EX_1 = 0$ when $1 < \alpha \leq 2$. If

$$\begin{aligned} &E|X_1|^{\alpha} < \infty, \quad \textit{for } \alpha > \gamma, \\ &E|X_1|^{\alpha} \log\bigl(1 + |X_1|\bigr) < \infty, \quad \textit{for } \alpha = \gamma, \\ &E|X_1|^{\gamma} < \infty, \quad \textit{for } \alpha < \gamma, \end{aligned} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} \frac{1}{n} P\left(\max_{1\leq j\leq n}\left|\sum_{i=1}^{j} a_{ni}X_i\right| > \varepsilon b_n\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.3}$$

Zhou *et al.* [30] partially extended Theorem 1.1 for negatively associated random variables to the case of $\tilde{\rho}$-mixing random variables as follows.

Theorem 1.2 *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed $\tilde{\rho}$-mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying*

$$\sum_{i=1}^{n} |a_{ni}|^{\max\{\alpha,\gamma\}} = O(n) \tag{1.4}$$

for some $0 < \alpha \leq 2$ and $\gamma > 0$ with $\alpha \neq \gamma$. Let $b_n = n^{1/\alpha} \log^{1/\gamma} n$. If $EX_1 = 0$ for $1 < \alpha \leq 2$ and (1.2) *holds for $\alpha \neq \gamma$, then* (1.3) *holds.*

Zhou *et al.* [30] left an open problem whether the case $\alpha = \gamma$ of Theorem 1.1 holds for $\tilde{\rho}$-mixing random variables. Sung [27] solved the open problem and obtained the following strong convergence result for weighted sums of $\tilde{\rho}$-mixing random variables.

Theorem 1.3 *Let $\{X_n, n \geq 1\}$ be a sequence of identically distributed $\tilde{\rho}$-mixing random variables, and let $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of constants satisfying* (1.1) *for some $0 < \alpha \leq 2$. Let $b_n = n^{1/\alpha} \log^{1/\alpha} n$. If $EX_1 = 0$ for $1 < \alpha \leq 2$ and $E|X_1|^\alpha \log(1 + |X_1|) < \infty$, then* (1.3) *holds.*

Sung [27] also left an open problem whether the case $\alpha < \gamma$ of Theorem 1.1 holds for $\tilde{\rho}$-mixing random variables. In this paper, we will partially solve the open problem using a different method from Zhou *et al.* [30] and Sung [27]. As an application, the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums of $\tilde{\rho}$-mixing random variables is obtained. The results presented in this paper are obtained by using the truncated method and the Rosenthal-type inequality of $\tilde{\rho}$-mixing random variables (Lemma 2.1 in Section 2).

Throughout the paper, let $I(A)$ be the indicator function of the set A. C denotes a positive constant, which may be different in various places, and $a_n = O(b_n)$ stands for $a_n \leq Cb_n$. Denote $\log x = \ln \max(x, e)$.

2 Main results

Firstly, let us recall the definition of stochastic domination which will be used frequently in the paper.

Definition 2.1 A sequence of random variables $\{X_n, n \geq 1\}$ is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P\big(|X_n| > x\big) \leq CP\big(|X| > x\big)$$

for all $x \geq 0$ and $n \geq 1$.

To prove the main results of the paper, we need the following two lemmas. The first one is the Rosenthal-type inequality for $\tilde{\rho}$-mixing random variables. The proof can be found in Utev and Peligrad [5].

Lemma 2.1 (Utev and Peligrad [5]) *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables with $EX_n = 0$, $E|X_n|^p < \infty$ for some $p \geq 2$ and each $n \geq 1$. Then there exists a positive* *constant C depending only on p such that*

$$E\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}X_i\right|^p\right)\le C\left\{\sum_{i=1}^{n}E|X_i|^p+\left(\sum_{i=1}^{n}EX_i^2\right)^{p/2}\right\}.$$

The next one is the basic property for stochastic domination. The proof is standard, so we omit it.

Lemma 2.2 *Let $\{X_n, n\ge 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold*:

$$E|X_n|^\alpha I(|X_n|\le b)\le C_1\left[E|X|^\alpha I(|X|\le b)+b^\alpha P(|X|>b)\right],$$
$$E|X_n|^\alpha I(|X_n|>b)\le C_2E|X|^\alpha I(|X|>b),$$

where C_1 and C_2 are positive constants. Consequently, $E|X_n|^\alpha \le CE|X|^\alpha$.

Our main results are as follows.

Theorem 2.1 *Let $\{X_n, n\ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, i\ge 1, n\ge 1\}$ be an array of constants. Assume that the following two conditions are satisfied*:

(i) *There exist some δ with $0<\delta<1$ and some α with $0<\alpha\le 2$ such that $\sum_{i=1}^n |a_{ni}|^\alpha = O(n^\delta)$, and assume further that $EX_n = 0$ when $1<\alpha\le 2$;*

(ii) *$p\ge 1/\alpha$. For some $\beta > \max\{p\alpha^2, \alpha+\frac{\alpha(p\alpha-1)}{1-\delta}, \alpha+2, \alpha(p\alpha-1)+2\delta\}$, $E|X|^\beta<\infty$.*

Then for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p\alpha-2}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon b_n\right)<\infty, \tag{2.1}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$.

Similar to the proof of Theorem 2.1 and by weakening the condition (i) of Theorem 2.1 (*i.e.*, $\sum_{i=1}^n |a_{ni}|^\alpha = O(n^\delta)$ is replaced by $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$), we can get the following strong convergence result for the special case $p\alpha = 1$. The proof is omitted.

Theorem 2.2 *Let $\{X_n, n\ge 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables which is stochastically dominated by a random variable X, and let $\{a_{ni}, i\ge 1, n\ge 1\}$ be an array of constants. Assume that there exists some α with $0<\alpha\le 2$ such that $\sum_{i=1}^n |a_{ni}|^\alpha = O(n)$, and assume further that $EX_n = 0$ when $1<\alpha\le 2$. If there exists some $\beta > \alpha+2$ such that $E|X|^\beta<\infty$, then for any $\varepsilon>0$,*

$$\sum_{n=1}^{\infty}\frac{1}{n}P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon b_n\right)<\infty, \tag{2.2}$$

where $b_n \doteq n^{1/\alpha}\log^{1/\gamma} n$ for some $\gamma > 0$.

If the array of constants $\{a_{ni}, i \geq 1, n \geq 1\}$ is replaced by a sequence of constants $\{a_n, n \geq 1\}$, then we can get the Marcinkiewicz-Zygmund type strong law of large numbers for weighted sums $S_n \doteq \sum_{i=1}^n a_i X_i$ of $\tilde{\rho}$-mixing sequence of random variables as follows.

Theorem 2.3 *Let $\{X_n, n \geq 1\}$ be a sequence of $\tilde{\rho}$-mixing random variables which is stochastically dominated by a random variable X, and let $\{a_n, n \geq 1\}$ be a sequence of constants. Assume that there exists some α with $0 < \alpha \leq 2$ such that $\sum_{i=1}^n |a_i|^\alpha = O(n)$, and assume further that $EX_n = 0$ when $1 < \alpha \leq 2$. If there exists some $\beta > \alpha + 2$ such that $E|X|^\beta < \infty$, then for any $\varepsilon > 0$,*

$$\sum_{n=1}^{\infty} \frac{1}{n} P\Big(\max_{1\leq j\leq n} |S_j| > \varepsilon b_n\Big) < \infty \tag{2.3}$$

and

$$\lim_{n\to\infty} \frac{S_n}{b_n} = 0 \quad a.s., \tag{2.4}$$

where $b_n \doteq n^{1/\alpha} \log^{1/\gamma} n$ for some $\gamma > 0$ and $S_n \doteq \sum_{i=1}^n a_i X_i$ for $n \geq 1$.

Remark 2.1 In Theorem 2.2, we not only consider the case $\alpha < \gamma$, but also consider the cases $\alpha > \gamma$ and $\alpha = \gamma$. The only defect is that our moment condition '$E|X|^\beta < \infty$ for some $\beta > \alpha + 2$' is stronger than the corresponding one of Theorem 1.1. So, our main result partially settles the open problem posed by Sung [27]. In addition, we extend the results of Zhou *et al.* [30] and Sung [27] for identically distributed $\tilde{\rho}$-mixing random variables to the case of non-identical distribution.

3 The proofs

Proof of Theorem 2.1 For fixed $n \geq 1$, define

$$X_i^{(n)} = X_i I\big(|X_i| \leq b_n\big), \quad i \geq 1, \qquad T_j^{(n)} = \sum_{i=1}^{j} a_{ni}\big(X_i^{(n)} - EX_i^{(n)}\big), \quad j = 1, 2, \ldots, n.$$

It is easy to check that for any $\varepsilon > 0$,

$$\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon b_n\right) \subset \Big(\max_{1\leq i\leq n} |X_i| > b_n\Big) \cup \left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i^{(n)}\right| > \varepsilon b_n\right),$$

which implies that

$$\begin{aligned} &P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i\right| > \varepsilon b_n\right) \\ &\quad\leq P\Big(\max_{1\leq i\leq n} |X_i| > b_n\Big) + P\left(\max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} X_i^{(n)}\right| > \varepsilon b_n\right) \\ &\quad\leq \sum_{i=1}^{n} P\big(|X_i| > b_n\big) + P\left(\max_{1\leq j\leq n} \big|T_j^{(n)}\big| > \varepsilon b_n - \max_{1\leq j\leq n} \left|\sum_{i=1}^{j} a_{ni} EX_i^{(n)}\right|\right). \end{aligned} \tag{3.1}$$

Firstly, we will show that

$$b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_i^{(n)}\right|\to 0\quad\text{as } n\to\infty.\tag{3.2}$$

By $\sum_{i=1}^{n}|a_{ni}|^\alpha=O(n^\delta)$ and Hölder's inequality, we have for $1\le k<\alpha$ that

$$\sum_{i=1}^{n}|a_{ni}|^k\le\left(\sum_{i=1}^{n}\left(|a_{ni}|^k\right)^{\frac{\alpha}{k}}\right)^{\frac{k}{\alpha}}\left(\sum_{i=1}^{n}1\right)^{\frac{\alpha-k}{\alpha}}\le Cn.\tag{3.3}$$

Hence, when $1<\alpha\le 2$, we have by $EX_n=0$, Lemma 2.2, (3.3) (taking $k=1$), Markov's inequality and condition (ii) that

$$\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_i^{(n)}\right| &= b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}EX_iI\big(|X_i|>b_n\big)\right|\\
&\le b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X_i|I\big(|X_i|>b_n\big)\\
&\le b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X|I\big(|X|>b_n\big)\\
&\le Cb_n^{-1}nE|X|I\big(|X|>b_n\big)\\
&= Cb_n^{-1}n\sum_{k=n}^{\infty}E|X|I\big(b_k<|X|\le b_{k+1}\big)\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}P\big(|X|>b_k\big)\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}b_{k+1}\frac{E|X|^\beta}{b_k^\beta}\\
&\le Cb_n^{-1}n\sum_{k=n}^{\infty}\frac{(k+1)^{1/\alpha}\log^{1/\gamma}(k+1)}{k^{\beta/\alpha}\log^{\beta/\gamma}k}\\
&\le Cb_n^{-1}\sum_{k=n}^{\infty}k^{1/\alpha+1-\beta/\alpha}\\
&\le \frac{Cn^{1/\alpha+2-\beta/\alpha}}{n^{1/\alpha}\log^{1/\gamma}n}\to 0\quad\text{as } n\to\infty.
\end{aligned}\tag{3.4}$$

Elementary Jensen's inequality implies that for any $0<s<t$,

$$\left(\sum_{i=1}^{n}|a_{ni}|^t\right)^{1/t}\le\left(\sum_{i=1}^{n}|a_{ni}|^s\right)^{1/s}.\tag{3.5}$$

Therefore, when $0 < \alpha \le 1$, we have by Lemma 2.2, (3.5), Markov's inequality and condition (ii) that

$$\begin{aligned}
b_n^{-1}\max_{1\le j\le n}\left|\sum_{i=1}^{j} a_{ni}EX_i^{(n)}\right| &\le b_n^{-1}\sum_{i=1}^{n}|a_{ni}|E|X_{ni}|I\big(|X_{ni}|\le b_n\big) \\
&\le Cb_n^{-1}\sum_{i=1}^{n}|a_{ni}|\big(E|X|I\big(|X|\le b_n\big)+b_nP\big(|X|>b_n\big)\big) \\
&\le Cb_n^{-1}n^{\delta/\alpha}E|X|I\big(|X|\le b_n\big)+Cn^{\delta/\alpha}P\big(|X|>b_n\big) \\
&\le Cb_n^{-1}n^{\delta/\alpha}E|X|I\big(|X|\le b_n\big)+Cn^{\delta/\alpha}P\big(|X|>b_n\big) \\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}E|X|I\big(b_{k-1}<|X|\le b_k\big)+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{b_n^{\beta}} \\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_kP\big(|X|>b_{k-1}\big)+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n} \\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}b_k\frac{E|X|^{\beta}}{b_{k-1}^{\beta}}+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n} \\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}\frac{k^{1/\alpha}\log^{1/\gamma}k}{(k-1)^{\beta/\alpha}\log^{\beta/\gamma}(k-1)}+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n} \\
&\le Cb_n^{-1}n^{\delta/\alpha}\sum_{k=2}^{n}k^{1/\alpha-\beta/\alpha}+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n} \\
&\le \frac{Cn^{\delta/\alpha}n^{1/\alpha-\beta/\alpha+1}}{n^{1/\alpha}\log^{1/\gamma}n}+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n} \\
&= \frac{C}{n^{\beta/\alpha-\delta/\alpha-1}\log^{1/\gamma}n}+\frac{Cn^{\delta/\alpha}E|X|^{\beta}}{n^{\beta/\alpha}\log^{\beta/\gamma}n}\to 0 \quad \text{as } n\to\infty. \qquad (3.6)
\end{aligned}$$

Equations (3.4) and (3.6) yield (3.2). Hence, for n large enough,

$$P\left(\max_{1\le j\le n}\left|\sum_{i=1}^{j}a_{ni}X_i\right|>\varepsilon b_n\right)\le\sum_{i=1}^{n}P\big(|X_i|>b_n\big)+P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}b_n\right).$$

To prove (2.1), we only need to show that

$$I \doteq \sum_{n=1}^{\infty}n^{p\alpha-2}\sum_{i=1}^{n}P\big(|X_i|>b_n\big)<\infty \qquad (3.7)$$

and

$$J \doteq \sum_{n=1}^{\infty}n^{p\alpha-2}P\left(\max_{1\le j\le n}\left|T_j^{(n)}\right|>\frac{\varepsilon}{2}b_n\right)<\infty. \qquad (3.8)$$

By the definition of stochastic domination, Markov's inequality and condition (ii), we can see that

$$\begin{aligned} I &\doteq \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\big(|X_i| > b_n\big) \\ &\le C \sum_{n=1}^{\infty} n^{p\alpha-2} \sum_{i=1}^{n} P\big(|X| > b_n\big) \le C \sum_{n=2}^{\infty} n^{p\alpha-1} \frac{E|X|^{\beta}}{b_n^{\beta}} \\ &\le C \sum_{n=2}^{\infty} \frac{n^{p\alpha-1}}{n^{\beta/\alpha} \log^{\beta/\gamma} n} < \infty \quad \big(\text{since } \beta > p\alpha^2\big). \end{aligned} \tag{3.9}$$

For $q \ge 2$, it follows from Lemma 2.1, C_r-inequality and Jensen's inequality that

$$\begin{aligned} J &\doteq \sum_{n=1}^{\infty} n^{p\alpha-2} P\left(\max_{1\le j\le n} \big|T_j^{(n)}\big| > \frac{\varepsilon}{2} b_n\right) \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} E\Big(\max_{1\le j\le n} \big|T_j^{(n)}\big|^q\Big) \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \left[\sum_{i=1}^{n} |a_{ni}|^q E\big|X_i^{(n)} - EX_i^{(n)}\big|^q + \left(\sum_{i=1}^{n} |a_{ni}|^2 E\big|X_i^{(n)} - EX_i^{(n)}\big|^2 \right)^{q/2} \right] \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q E\big|X_i^{(n)}\big|^q \\ &\quad + C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \left(\sum_{i=1}^{n} |a_{ni}|^2 E\big|X_i^{(n)}\big|^2 \right)^{q/2} \\ &\doteq J_1 + J_2. \end{aligned} \tag{3.10}$$

Take a suitable constant q such that $\max\{2, \alpha(p\alpha-1)/(1-\delta)\} < q < \min\{\beta-\alpha, \frac{\beta-p\alpha^2+\alpha}{\delta}\}$, which implies that

$$\beta > \alpha + q, \qquad \beta/\alpha - q/\alpha > 1, \qquad \beta > p\alpha^2 - \alpha + q\delta, \qquad \beta/\alpha - p\alpha + 2 - q\delta/\alpha > 1$$

and

$$p\alpha - 2 + q\delta/\alpha - q/\alpha < -1, \quad q > \alpha.$$

It follows from Lemma 2.2, (3.5), Markov's inequality and condition (ii) that

$$\begin{aligned} J_1 &\doteq C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q E\big|X_i^{(n)}\big|^q \\ &= C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q E|X_i|^q I\big(|X_i| \le b_n\big) \\ &\le C \sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \sum_{i=1}^{n} |a_{ni}|^q \big[E|X|^q I\big(|X| \le b_n\big) + b_n^q P\big(|X| > b_n\big)\big] \end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} E|X|^q I\bigl(|X|\le b_n\bigr) + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} P\bigl(|X|>b_n\bigr) \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} \sum_{k=2}^{n} E|X|^q I\bigl(b_{k-1}<|X|\le b_k\bigr) + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} \frac{E|X|^\beta}{b_n^\beta} \\
&\le C\sum_{k=2}^{\infty}\sum_{n=k}^{\infty} n^{p\alpha-2+q\delta/\alpha-q/\alpha} (\log n)^{-q/\gamma} b_k^q P\bigl(|X|>b_{k-1}\bigr) + C\sum_{n=2}^{\infty} \frac{n^{p\alpha-2+q\delta/\alpha}}{n^{\beta/\alpha}\log^{\beta/\gamma} n} \\
&\le C\sum_{k=3}^{\infty} b_k^q \frac{E|X|^\beta}{b_{k-1}^\beta} + C\sum_{n=2}^{\infty} \frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma} n} \\
&\le C\sum_{k=3}^{\infty} \frac{k^{q/\alpha}\log^{q/\gamma} k}{(k-1)^{\beta/\alpha}\log^{\beta/\gamma}(k-1)} + C\sum_{n=2}^{\infty} \frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma} n} \\
&\le C\sum_{k=3}^{\infty} \frac{1}{k^{\beta/\alpha-q/\alpha}} + C\sum_{n=2}^{\infty} \frac{1}{n^{\beta/\alpha-p\alpha+2-q\delta/\alpha}\log^{\beta/\gamma} n} < \infty.
\end{aligned}
\tag{3.11}
$$

By Lemma 2.2 again, (3.5), C_r-inequality and Jensen's inequality, we can get that

$$
\begin{aligned}
J_2 &\doteq C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \left(\sum_{i=1}^{n} |a_{ni}|^2 E\bigl|X_i^{(n)}\bigr|^2\right)^{q/2} \\
&= C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \left(\sum_{i=1}^{n} |a_{ni}|^2 E|X_i|^2 I\bigl(|X_i|\le b_n\bigr)\right)^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2} b_n^{-q} \left[\sum_{i=1}^{n} |a_{ni}|^2 \bigl(EX^2 I\bigl(|X|\le b_n\bigr) + b_n^2 P\bigl(|X|>b_n\bigr)\bigr)\right]^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} \bigl[EX^2 I\bigl(|X|\le b_n\bigr) + b_n^2 P\bigl(|X|>b_n\bigr)\bigr]^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} \bigl[EX^2 I\bigl(|X|\le b_n\bigr)\bigr]^{q/2} + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} \bigl[P\bigl(|X|>b_n\bigr)\bigr]^{q/2} \\
&\le C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} b_n^{-q} E|X|^q I\bigl(|X|\le b_n\bigr) + C\sum_{n=2}^{\infty} n^{p\alpha-2+q\delta/\alpha} P\bigl(|X|>b_n\bigr) \\
&< \infty \quad \text{(similar to the proof of (3.11))}.
\end{aligned}
\tag{3.12}
$$

Therefore, the desired result (2.1) follows from (3.9)-(3.12) immediately. This completes the proof of the theorem. □

Proof of Theorem 2.3 Similar to the proof of Theorem 2.1, we can get (2.3) immediately. Therefore,

$$
\begin{aligned}
\infty &> \sum_{n=1}^{\infty} \frac{1}{n} P\Bigl(\max_{1\le j\le n} |S_j| > \varepsilon b_n\Bigr) \\
&= \sum_{i=0}^{\infty}\sum_{n=2^i}^{2^{i+1}-1} \frac{1}{n} P\Bigl(\max_{1\le j\le n} |S_j| > \varepsilon n^{\frac{1}{\alpha}} (\log n)^{\frac{1}{\gamma}}\Bigr) \ge \frac{1}{2}\sum_{i=1}^{\infty} P\Bigl(\max_{1\le j\le 2^i} |S_j| > \varepsilon 2^{\frac{i+1}{\alpha}} \bigl(\log 2^{i+1}\bigr)^{\frac{1}{\gamma}}\Bigr).
\end{aligned}
$$

By the Borel-Cantelli lemma, we obtain that

$$\lim_{i\to\infty} \frac{\max_{1\le j\le 2^i} |S_j|}{2^{\frac{i+1}{\alpha}} (\log 2^{i+1})^{\frac{1}{\gamma}}} = 0 \quad \text{a.s.} \tag{3.13}$$

For all positive integers n, there exists a positive integer i_0 such that $2^{i_0-1} \le n < 2^{i_0}$. We have by (3.13) that

$$\frac{|S_n|}{b_n} \le \max_{2^{i_0-1}\le n<2^{i_0}} \frac{|S_n|}{b_n} \le \frac{2^{\frac{2}{\alpha}} \max_{1\le j\le 2^{i_0}} |S_j|}{2^{\frac{i_0+1}{\alpha}} (\log 2^{i_0+1})^{\frac{1}{\gamma}}} \left(\frac{i_0+1}{i_0-1}\right)^{\frac{1}{\gamma}} \to 0 \quad \text{a.s. as } i_0 \to \infty,$$

which implies (2.4). This completes the proof of the theorem. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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