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The maximal and minimal ranks of matrix expression with applications

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Abstract

We give in this article the maximal and minimal ranks of the matrix expression $A-B_1V_1C_1-B_2V_2C_2-B_3V_3C_3-B_4V_4C_4$ with respect to V_1, V_2, V_3, and V_4. As applications, we derive the extremal ranks of the generalized Schur complement $A - BM^{(1)}C - DN^{(1)}G$ and the partial matrix $(A\ BM^{(1)}C\ DN^{(1)}G)$ with respect to the generalized inverse $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$.

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Keywords: matrix expression, maximal rank, minimal rank, generalized inverse, rank equality

1 Introduction

Let $C^{m \times n}$ be the set of all $m \times n$ complex matrices with complex entries. I_n denotes the identity matrix of order n and $O_{m \times n}$ denotes the $m \times n$ matrix of all zero entries (if no confusion occurs, we will omit the subscript). For a given a matrix $A \in C^{m \times n}$, the symbols A^* and $r(A)$ will stand for the conjugate transpose and the rank of the matrix A, respectively. We recall that a generalized inverse $X \in C^{n \times m}$ of $A \in C^{m \times n}$ is a matrix which satisfies some of the following four Penrose equations [1]:

$$(1)\,AXA = A, \quad (2)\,XAX = X, \quad (3)\,(AX)^* = AX, \quad (4)\,(XA)^* = XA.$$

For a subset $\{i,j,\ldots,k\}$ of the set $\{1,2,3,4\}$, the set of $n \times m$ matrices satisfying the equations (i), (j), ...,(k) from among the above four Penrose Equations (1)-(4) is denoted by $A\{i,j,\ldots,k\}$. A matrix X from $A\{i,j,\ldots,k\}$ is called an $\{i,j,\ldots,k\}$-inverse of A and is denoted by $A^{(i,j,\ldots,k)}$. In particular, an $n \times m$ matrix X of the set $A\{1\}$ is called a g-inverse of A and denoted by $A^{(1)}$. The unique $\{1, 2, 3, 4\}$-inverse of A is denoted by $A^\dagger$, which is called the Moore-Penrose inverse of A. Throughout this article, the abbreviated symbols E_A and F_A stand for the two projectors $E_A = I - AA^\dagger$ and $F_A = I - A^\dagger A$ of A, respectively. We refer the reader to [2,3] for basic results on the generalized inverses.

Given a matrix with some variant entries in it (often called partial matrix) or a matrix expression with some variant matrices in it, the rank of the partial matrix or matrix expression will vary with respect to the variant entries or variant matrices. Because the rank of matrix is an integer between 0 and the minimal of row and column numbers of the matrix, maximal and minimal ranks of partial matrix or matrix expressions must exist with respect to their variant entries or variant matrices. Many problems in matrix theory and applications are closely related to maximal and minimal possible ranks of matrix expressions with variant entries. For example, a matrix equation $AXB = C$ is consistent if and only if the minimal rank of C-AXB with respect to X is zero, see [4-6]; there is matrix X such that the partial matrix AXB of order n is nonsingular if and only if the maximal rank of AXB with respect to X is n, see [7-11].

The maximal and minimal ranks of matrix expressions or partial matrix are two basic concepts in matrix theory for describing the dimension of the row or column vector space of matrix expressions or partial matrix, both of which are well understood and are easy to compute by the well-known elementary or congruent matrix operations, see [5,7,8,10-16]. These two quantities play an essential role in characterizing algebraic properties of matrices expressions or partial matrices. In fact, maximal and minimal ranks of matrix expressions or partial matrices have been the main objects of study in matrix theory and applications. Some previous systematical researches on maximal and minimal ranks of matrix expressions or partial matrices and their applications can be found in [17-20]. In recent years, the present author reconsidered the maximal and minimal ranks of matrix expressions or partial matrices by using some tricky operations on block matrices and generalized inverses of matrices, and obtained many new formulas for maximal and minimal ranks of matrix expressions or partial matrices and their applications, see [4,6,9,21-28].

In this article, given matrices $A \in C^{m\times n},\ B_i \in C^{m\times p_i},\ C_i \in C^{q_i\times n},\ i = 1, 2, 3, 4$, we will present the maximal and minimal ranks of the matrix expression A-$B_1V_1C_1$ - $B_2V_2C_2$ - $B_3V_3C_3$ - $B_4V_4C_4$ with respect to V_1, V_2, V_3, and V_4. As applications, the maximal and minimal ranks of the generalized Schur complement A - $BM^{(1)}C$ - $DN^{(1)}G$ and the partial matrix (A $BM^{(1)}C$ $DN^{(1)}G$) with respect to the generalized inverse $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$ are also considered. The results in this article extend the earlier studies by various authors, see, e.g., [4-6,11,16,18,21,25,26].

We first introduce some well-known results which will be used in this article.

Lemma 1.1 [5,8,25]. Let

$$M = \begin{pmatrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix}$$

where $A_{ij} \in C^{m_i\times n_j}(1 \le i,\ j \le 3)$ are given, $X \in C^{m_1\times n_3}$ and $Y \in C^{m_3\times n_1}$ are two variant matrices. Then

$$\max_{X,\,Y} r(M) = \min\left\{m_3 + n_3 + r\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},\ m_1 + n_1 + r\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix},\right.$$
$$\left. m_1 + m_3 + r(A_{21}\ A_{22}\ A_{23}),\ n_1 + n_3 + r\begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix}\right\}, \tag{1}$$

$$\min_{X,\,Y} r(M) = r(A_{21}\ \ A_{22}\ A_{23}) + r\begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix} + \max\left\{r\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - r\begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} - r(A_{21}\ A_{22}),\right.$$
$$\left. r\begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - r\begin{pmatrix} A_{22} \\ A_{32} \end{pmatrix} - r(A_{22}\ A_{23})\right\}. \tag{2}$$

Lemma 1.2 [2]. Let $A \in C^{m\times n}$. Then the expression of {1}-inverses of A can be written as

$$A^{(1)} = A^{\dagger} + (I_n - A^{\dagger}A)W + Z(I_m - AA^{\dagger}), \tag{3}$$

where $W \in C^{n \times m}$ and $Z \in C^{n \times m}$ are arbitrary.

Lemma 1.3 [9]. Let $A \in C^{m \times n}$, $B \in C^{m \times k}$, and $C \in C^{l \times n}$. Then

$$(1).\ r(A\ B) = r(A) + r(E_A B) = r(E_B A) + r(B),$$

$$(2).\ r\begin{pmatrix} A \\ C \end{pmatrix} = r(A) + r(CF_A) = r(AF_C) + r(C),$$

where $E_A = I_m - AA^{\dagger}$ and $F_A = I_n - A^{\dagger}A$.

2 The maximal and minimal ranks of $A - B_1V_1C_1 - B_2V_2C_2 - B_3V_3C_3 - B_4V_4C_4$

In this section, we will present the maximal and minimal ranks of the linear matrix expression

$$P(V_1, V_2, V_3, V_4) = A - B_1V_1C_1 - B_2V_2C_2 - B_3V_3C_3 - B_4V_4C_4, \tag{4}$$

where $A \in C^{m\times n}$, $B_i \in C^{m\times p_i}$, $C_i \in C^{q_i\times n}$, $i = 1, 2, 3, 4$, are given matrices, with respect to four variant matrices $V_i \in C^{p_i\times q_i}$, $i = 1, 2, 3, 4$. Applying the formula (1) in Lemma 1.1 to the linear matrix expression in (4) and simplifying, we obtain the following result.

Theorem 2.1 Let $P(V_1, V_2, V_3, V_4)$ be given as (4). Then

$$\max_{V_1, V_2, V_3, V_4} r(P(V_1,\ V_2,\ V_3,\ V_4)) = \min\{T_1, T_2, T_3, T_4\}, \tag{5}$$

where

$$r(P(V_1, V_2, V_3, V_4)) = r\begin{pmatrix} O & O & O & O & O & O & O & I_{P_4} & -V_4 \\ O & O & O & O & C_4 & O & O & O & I_{q_4} \\ O & O & O & O & O & I_{P_2} & -V_2 & O & O \\ O & O & O & O & C_2 & O & I_{q_2} & O & O \\ O & B_3 & O & B_1 & A & B_2 & O & B_4 & O \\ O & O & I_{q_1} & O & C_1 & O & O & O & O \\ O & O & -V_1 & I_{P_1} & O & O & O & O & O \\ I_{q_3} & O & O & O & C_3 & O & O & O & O \\ -V_3 & I_{P_3} & O & O & O & O & O & O & O \end{pmatrix} - \sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i,$$

$$= r(T) - \sum_{i=1}^{4} pi - \sum_{i=1}^{4} qi,$$

Proof. It is easy to verify by block Gaussian elimination that the rank of $P(V_1,V_2,V_3, V_4)$ in (4) can be expressed as

$$r(P(V_1, V_2, V_3, V_4)) = r\begin{pmatrix} O & O & O & O & O & O & O & I_{P4} & -V_4 \\ O & O & O & O & C_4 & O & O & O & I_{q4} \\ O & O & O & O & O & I_{P2} & -V_2 & O & O \\ O & O & O & O & C_2 & O & I_{q2} & O & O \\ O & B_3 & O & B_1 & A & B_2 & O & B_4 & O \\ O & O & I_{q1} & O & C_1 & O & O & O & O \\ O & O & -V_1 & I_{P1} & O & O & O & O & O \\ I_{q3} & O & O & O & C_3 & O & O & O & O \\ -V_3 & I_{P3} & O & O & O & O & O & O & O \end{pmatrix} - \sum_{i=1}^{4} pi - \sum_{i=1}^{4} qi,$$

$$= r(T) - \sum_{i=1}^{4} pi - \sum_{i=1}^{4} qi,$$

where $A \in C^{m\times n},\ B_i \in C^{m\times p_i},\ C_i \in C^{q_i\times n},\ V_i \in C^{p_i\times q_i},\ i = 1, 2, 3, 4$ and $I_{p_i}, I_{q_i}, i = 1, 2, 3, 4$, are denotes the identity matrix of order p_i and q_i, respectively.

$$T = \begin{pmatrix} O & E_2 & -V_4 \\ E_1 & S & E_3 \\ -V_3 & E_4 & O \end{pmatrix}, S = \begin{pmatrix} O & O & O & C_4 & O & O & O \\ O & O & O & O & I_{p_2} & -V_2 & O \\ O & O & O & C_2 & O & I_{q_2} & O \\ B_3 & O & B_1 & A & B_2 & O & B_4 \\ O & I_{q_1} & O & C_1 & O & O & O \\ O & -V_1 & I_{p_1} & O & O & O & O \\ O & O & O & C_3 & O & O & O \end{pmatrix}$$

and

$$E_1 = (O\,O\,O\,O\,O\,O\,I_{q_3})^*,\quad E_2 = (O\,O\,O\,O\,O\,O\,I_{p_4}),$$
$$E_3 = (I_{q_4}\,O\,O\,O\,O\,O\,O)^*,\quad E_4 = (I_{p_3}\,O\,O\,O\,O\,O\,O).$$

According to this result, we have

$$\max_{V_1, V_2, V_3, V_4} r(P(V_1, V_2, V_3, V_4)) = \max_{V_1, V_2, V_3, V_4} r(T) - \sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i. \tag{6}$$

Then applying the formula (1) in Lemma 1.1 to matrix T, we have

$$\begin{aligned}\max_{V_3, V_4} r(T) &= \min\left\{p_3 + q_4 + r\begin{pmatrix} O & E_2 \\ E_1 & S \end{pmatrix} p_4 + q_3 + r\begin{pmatrix} S & E_3 \\ E_4 & O \end{pmatrix},\right.\\ &\qquad \left. p_4 + p_3 + r(E_1\ S\ E_2),\ q_3 + q_4 + r\begin{pmatrix} E_2 \\ S \\ E_4 \end{pmatrix}\right\}\\ &= \min\{p_3 + q_4 + p_4 + q_3 + r(S_1),\ p_4 + q_3 + q_4 + p_3 + r(S_2),\\ &\qquad p_4 + p_3 + q_4 + q_3 + r(S_3),\ q_3 + q_4 + p_4 + p_3 + r(S_4)\},\end{aligned}$$

where

$$S_1 = \begin{pmatrix} O & O & O & O & I_{P_2} & -V_2 \\ O & O & O & C_4 & O & O \\ O & O & O & C_2 & O & I_{q_2} \\ O & B_3 & B_1 & A & B_2 & O \\ I_{q_1} & O & O & C_1 & O & O \\ -V_1 & O & I_{p_1} & O & O & O \end{pmatrix}, S_2 = \begin{pmatrix} O & O & O & I_{P_2} & O & -V_2 \\ O & O & C_2 & O & O & I_{q_2} \\ O & B_1 & A & B_2 & B_4 & O \\ I_{q_1} & O & C_1 & O & O & O \\ O & O & C_3 & O & O & O \\ -V_1 & I_{p_1} & O & O & O & O \end{pmatrix}$$

$$S_3 \begin{pmatrix} O & O & O & O & I_{P_2} & O & -V_2 \\ O & O & O & C_2 & O & O & I_{q_2} \\ O & B_3 & B_1 & A & B_2 & B_4 & O \\ I_{q_1} & O & O & C_1 & O & O & O \\ -V_1 & O & I_{p_1} & O & O & O & O \end{pmatrix}, S_4 = \begin{pmatrix} O & O & O & I_{P_2} & -V_2 \\ O & O & C_4 & O & O \\ O & O & C_2 & O & I_{q_2} \\ O & B_1 & A & B_2 & O \\ I_{q_1} & O & C_1 & O & O \\ O & O & C_3 & O & O \\ -V_1 & I_{p_1} & O & O & O \end{pmatrix}$$

Again applying the formula (1) in Lemma 1.1, we get

$$\begin{aligned}\max_{V_1, V_2, V_3, V_4} r(T) = \min&\left\{p_3 + q_4 + p_4 + q_3 + \max_{V_1, V_2} r(S_1),\ p_4 + q_3 + q_4 + p_3 + \max_{V_1, V_2} r(S_2),\right.\\ &\left. p_4 + p_3 + q_4 + q_3 + \max_{V_1, V_2} r(S_3),\ q_3 + q_4 + p_4 + p_3 + \max_{V_1, V_2} r(S4)\right\}\end{aligned} \tag{7}$$

and

$$\max_{V_1, V_2} r(S_1) = \min\left\{p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_1 & A \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \\ O & C_1 & O \end{pmatrix},\right.$$
$$\left. p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & O & C_4 & O \\ B_3 & B_1 & A & B_2 \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_4 \\ O & C_2 \\ B_3 & A \\ O & C_1 \end{pmatrix}\right\}, \tag{8}$$

$$\max_{V_1, V_2} r(S_2) = \min\left\{p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} A & B_2 & B_4 \\ C_1 & O & O \\ C_3 & O & O \end{pmatrix},\right.$$
$$\left. p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} B_1 & A & B_2 & B_4 \\ O & C_3 & O & O \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} C_2 & O \\ A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix}\right\}, \tag{9}$$

$$\begin{aligned} \max_{V_1, V_2} r(S_3) = \min\Bigg\{ & p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & O & C_2 & O \\ B_3 & B_1 & A & B_4 \end{pmatrix}, \\ & p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} B_3 & A & B_2 & B_4 \\ O & C_1 & O & O \end{pmatrix}, \\ & p_1 + q_2 + q_1 + p_2 + r(B_3\ B_1\ A\ B_2\ B_4), \\ & p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_2 & O \\ B_3 & A & B_4 \\ O & C_1 & O \end{pmatrix}, \end{aligned} \tag{10}$$

$$\max_{V_1, V_2} r(S_4) = \min\left\{p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_4 \\ O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} C_4 & O \\ A & B_2 \\ C_1 & O \\ C_3 & O \end{pmatrix},\right.$$
$$p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} O & C_4 & O \\ B_1 & A & B_2 \\ O & C_3 & O \end{pmatrix}, p_1 + q_2 + q_1 + p_2 + r\begin{pmatrix} C_4 \\ C_2 \\ A \\ C_1 \\ C_3 \end{pmatrix}. \tag{11}$$

Substituting (8)-(11) into (7) and (6) yield (5).

Recall a simple fact that a matrix equation $AXB = C$ is consistent for every variant matrices X, if and only if the maximal rank of $C - AXB$ with respect to X is zero. Thus, by Theorem 2.1 we can immediately obtain the following result.

Corollary 2.2 Let $P(V_1,V_2,V_3, V_4)$ be given as (4). Then the matrix equation $A = B_1V_1C_1 + B_2V_2C_2 + B_3V_3C_3 + B_4V_4C_4$ holds for any V_1, V_2, V_3, and V_4 if and only if $T_1 = O$ or $T_2 = O$ or $T_3 = O$ or $T_4 = O$.

Because the right side of (5) are just composed by ranks of block matrices, they can be easily simplified by block Gaussian elimination when the given matrices in (4) satisfy some restrictions.

Theorem 2.3 Let $P(V_1, V_2, V_3, V_4)$ be given as (4) and let $R(B_1) \subseteq R(B_2)$, $R(B_3) \subseteq R(B_4)$, $R(B_1) \subseteq R(B_2), R(B_3) \subseteq R(B_4), R(C_2^*) \subseteq R(C_1^*), R(C_4^*) \subseteq R(C_3^*)$ Then

$$\max_{V_1, V_2, V_3, V_4} r(P(V_1,\ V_2,\ V_3,\ V_4)) = \min\{\tau_1,\ \tau_2,\ \tau_3\}, \tag{12}$$

where

$$\tau_1 = \min\left\{ r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_1 & A \end{pmatrix},\ r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \end{pmatrix},\ r\begin{pmatrix} O & C_4 \\ B_3 & A \\ O & C_1 \end{pmatrix} \right\},$$

$$\tau_2 = \min\left\{ r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix},\ r(A\ B_2\ B_4),\ r\begin{pmatrix} A & B_4 \\ C_1 & O \end{pmatrix} \right\},$$

$$\tau_3 = \min\left\{ r\begin{pmatrix} O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix},\ r\begin{pmatrix} A & B_2 \\ C_3 & O \end{pmatrix},\ r\begin{pmatrix} A \\ C_1 \\ C_3 \end{pmatrix} \right\}.$$

Proof. In fact, we can write $B_1 = B_2X$, $B_3 = B_4Y$, $C_2 = ZC_1$, and $C_4 = WC_3$ under the hypotheses of Theorem 2.3. In this case, we have

$$\begin{aligned} & r\begin{pmatrix} O & O & C_4 & O \\ B_3 & B_1 & A & B_2 \end{pmatrix} = r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \end{pmatrix},\ r\begin{pmatrix} O & C_4 \\ O & C_2 \\ B_3 & A \\ O & C_1 \end{pmatrix} = r\begin{pmatrix} O & C_4 \\ B_3 & A \\ O & C_1 \end{pmatrix}, \\ & r\begin{pmatrix} O & O & O \\ O & O & C_2 \\ B_3 & B_1 & A \end{pmatrix} = r\begin{pmatrix} O & O & C_4 \\ O & O & ZC_1 \\ B_3 & B_2X & A \end{pmatrix} \le r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \\ O & C_1 & O \end{pmatrix} \end{aligned} \tag{13}$$

and

$$\begin{aligned} & r\begin{pmatrix} B_1 & A & B_2 & B_4 \\ O & C_3 & O & O \end{pmatrix} = r\begin{pmatrix} A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix},\ r\begin{pmatrix} C_2 & O \\ A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} = r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix}, \\ & r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix} = r\begin{pmatrix} O & ZC_1 & O \\ B_2X & A & B_4 \\ O & C_3 & O \end{pmatrix} \le r\begin{pmatrix} A & B_2 & B_4 \\ C_1 & O & O \\ C_3 & O & O \end{pmatrix} \end{aligned} \tag{14}$$

and

$$\begin{aligned} & r(B_3\ B_1\ A\ B_2\ B_4) = r(A\ B_2\ B_4) = r\begin{pmatrix} O & C_2 & O \\ B_3 & A & B_4 \\ O & C_1 & O \end{pmatrix} = r\begin{pmatrix} A & B_4 \\ C_1 & O \end{pmatrix}, \\ & r\begin{pmatrix} O & O & C_2 & O \\ B_3 & B_1 & A & B_4 \end{pmatrix} = r\begin{pmatrix} O & O & ZC_1 & O \\ B_3 & B_2X & A & B_4 \end{pmatrix} \le r\begin{pmatrix} B_3 & A & B_2 & B_4 \\ O & C_1 & O & O \end{pmatrix}, \\ & r\begin{pmatrix} O & O & C_2 & O \\ B_3 & B_1 & A & B_4 \end{pmatrix} = r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix} \end{aligned} \tag{15}$$

and

$$\begin{aligned} & r\begin{pmatrix} O & C_4 & O \\ B_1 & A & B_2 \\ O & C_3 & O \end{pmatrix} = r\begin{pmatrix} A & B_2 \\ C_3 & O \end{pmatrix}, r\begin{pmatrix} C_4 \\ C_2 \\ A \\ C_1 \\ C_3 \end{pmatrix} = r\begin{pmatrix} A \\ C_1 \\ C_3 \end{pmatrix}, r\begin{pmatrix} O & C_4 \\ O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix} = r\begin{pmatrix} O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix}, \\ & r\begin{pmatrix} O & C_4 \\ O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix} = r\begin{pmatrix} O & C_4 \\ O & ZC_1 \\ B_2X & A \\ O & C_3 \end{pmatrix} \le r\begin{pmatrix} C_4 & O \\ A & B_2 \\ C_1 & O \\ C_3 & O \end{pmatrix} \end{aligned} \tag{16}$$

Combining (5) with (13)-(16) yields (12).

Corollary 2.4 Let $P(V_1, V_2, V_3, V_4)$ be given as (4) and let $R(B_1) \subseteq R(B_2), R(B_3) \subseteq R(B_4), R(C_2^*) \subseteq R(C_1^*), R(C_4^*) \subseteq R(C_3^*)$ then the matrix equation $A = B_1V_1C_1 + B_2V_2C_2 + B_3V_3C_3 + B_4V_4C_4$ holds for any V_1, V_2, V_3, and V_4 if and only if $\tau_1 = O$ or $\tau_2 = O$ or $\tau_3 = O$.

In the rest of this section, we will find the minimal rank of the linear matrix expression $P(V_1,V_2, V_3, V_4)$ in (4), with respect to four variant matrices $V_i \in C^{p_i \times q_i}$, $i = 1, 2, 3, 4$, when $P(V_1, V_2, V_3, V_4)$ satisfy some restrictions.

Theorem 2.5 Let $P(V_1, V_2, V_3, V_4)$ be given as (4) and let $R(B_1) \subseteq R(B_2), R(B_3) \subseteq R(B_4), R(C_2^*) \subseteq R(C_1^*), R(C_4^*) \subseteq R(C_3^*)$. Then

$$\begin{aligned}
&\min_{V_1, V_2, V_3, V_4} r(P(V_1, V_2, V_3, V_4)) \\
&= r(A\ B_2\ B_4) + r\begin{pmatrix} A & B_4 \\ C_1 & O \end{pmatrix} + r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix} + r\begin{pmatrix} A & B_2 \\ C_3 & O \end{pmatrix} + r\begin{pmatrix} O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix} \\
&+ r\begin{pmatrix} A \\ C_1 \\ C_3 \end{pmatrix} - r\begin{pmatrix} B_1 & A & B_4 \\ O & C_1 & O \end{pmatrix} - r\begin{pmatrix} O & C_2 & O \\ B_2 & A & B_4 \end{pmatrix} - r\begin{pmatrix} B_1 & A \\ O & C_1 \\ O & C_3 \end{pmatrix} - r\begin{pmatrix} C_2 & O \\ A & B_2 \\ C_3 & O \end{pmatrix} \\
&+ \max\left\{ r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \end{pmatrix} + r\begin{pmatrix} O & C_4 \\ B_3 & A \\ O & C_1 \end{pmatrix} + r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_1 & A \end{pmatrix} - r\begin{pmatrix} O & O & C_4 \\ O & O & C_1 \\ B_3 & B_1 & A \end{pmatrix} \right. \\
&- r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_2 & A \end{pmatrix} - \beta_1 - \beta_2,\ r\begin{pmatrix} A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix} + r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} \\
&\left. + r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix} - r\begin{pmatrix} B_1 & A & B_4 \\ O & C_1 & O \\ O & C_3 & O \end{pmatrix} - r\begin{pmatrix} C_2 & O & O \\ A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix} - 2\beta_3 \right\},
\end{aligned} \tag{17}$$

where

$$\beta_1 = \min\left\{ r\begin{pmatrix} O & O & C_2 \\ B_3 & B_1 & A \\ O & O & C_3 \end{pmatrix},\ r\begin{pmatrix} B_3 & A & B_2 \\ O & C_3 & O \end{pmatrix},\ r\begin{pmatrix} B_3 & A \\ O & C_1 \\ O & C_3 \end{pmatrix} \right\},$$

$$\beta_2 = \min\left\{ r\begin{pmatrix} O & C_4 & O \\ O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix},\ r\begin{pmatrix} C_4 & O & O \\ A & B_2 & B_4 \end{pmatrix},\ r\begin{pmatrix} C_4 & O \\ A & B_4 \\ C_1 & O \end{pmatrix} \right\},$$

$$\beta_3 = \min\left\{ r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix},\ r\begin{pmatrix} A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix},\ r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} \right\}.$$

Proof. From the proof of Theorem 2.1, it is easy to verify that the minimal rank of $P(V_1, V_2, V_3, V_4)$ in (4) can be expressed as

$$\min_{V_1, V_2, V_3, V_4} r(P(V_1, V_2, V_3, V_4)) = \min_{V_1, V_2, V_3, V_4} r(T) - \sum_{i=1}^{4} p_i - \sum_{i=1}^{4} q_i, \tag{18}$$

where T, S, E_i, p_i and q_i, $i = 1,2,3,4$, are given as the proof of Theorem 2.1. Then applying the formula (2) in Lemma 1.1 to matrix T, we have

$$\begin{aligned}\min_{V_1,V_2,V_3,V_4} r(T) &= \min_{V_3,V_4} r\begin{pmatrix} O & E_2 & -V_4 \\ E_1 & S & E_3 \\ -V_3 & E_4 & O \end{pmatrix} = r(E_1 \quad S \quad E_3) + r\begin{pmatrix} E_2 \\ S \\ E_4 \end{pmatrix} \\ &+ \max\left\{ r\begin{pmatrix} O & E_2 \\ E_1 & S \end{pmatrix} - r\begin{pmatrix} E_2 \\ S \end{pmatrix} - r(E_1 \quad S), \right. \\ &\left. r\begin{pmatrix} S & E_3 \\ E_4 & O \end{pmatrix} - r\begin{pmatrix} S \\ E_4 \end{pmatrix} - r(S \quad E_3) \right\}.\end{aligned} \tag{19}$$

In this case, we derive from (19) that

$$\begin{aligned}\min_{V_1,V_2,V_3,V_4} r(T) &= \min_{V_1,V_2} r(E_1 \quad S \quad E_3) + \min_{V_1,V_2} r\begin{pmatrix} E_2 \\ S \\ E_4 \end{pmatrix} \\ &+ \max\left\{ \min_{V_1,V_2} r\begin{pmatrix} O & E_2 \\ E_1 & S \end{pmatrix} - \min_{V_1,V_2} r\begin{pmatrix} E_2 \\ S \end{pmatrix} - \min_{V_1,V_2} r(E_1 \quad S), \right. \\ &\left. \min_{V_1,V_2} r\begin{pmatrix} S & E_3 \\ E_4 & O \end{pmatrix} - \min_{V_1,V_2} r\begin{pmatrix} S \\ E_4 \end{pmatrix} - \min_{V_1,V_2} r(S \quad E_3) \right\}.\end{aligned} \tag{20}$$

Again applying the formula (2) in Lemma 1.1, we have

$$\begin{aligned}&\min_{V_1,V_2} r(E_1 \quad S \quad E_3) = q_4 + q_3 + \min_{V_1,V_2} r(S_3) \\ &= \sum_{i=1}^{4} q_i + p_1 + p_2 + r(B_3 \quad B_1 \quad A \quad B_2 \quad B_4) + r\begin{pmatrix} O & C_2 & O \\ B_3 & A & B_4 \\ O & C_1 & O \end{pmatrix} \\ &+ \max\left\{ r\begin{pmatrix} O & O & C_2 & O \\ B_3 & B_1 & A & B_4 \end{pmatrix} - r\begin{pmatrix} O & O & C_2 & O \\ B_3 & B_1 & A & B_4 \\ O & O & C_1 & O \end{pmatrix} - r\begin{pmatrix} O & O & C_2 & O & O \\ B_3 & B_1 & A & B_2 & B_4 \end{pmatrix}, \right. \\ &\left. r\begin{pmatrix} B_3 & A & B_2 & B_4 \\ O & C_1 & O & O \end{pmatrix} - r\begin{pmatrix} O & C_2 & O & O \\ B_3 & A & B_2 & B_4 \\ O & C_1 & O & O \end{pmatrix} - r\begin{pmatrix} B_3 & B_1 & A & B_2 & B_4 \\ O & O & C_1 & O & O \end{pmatrix} \right\},\end{aligned} \tag{21}$$

where S_3 is given as the Equation (7) of the proof of Theorem 2.1. Since $B_1 = B_2X$, $B_3 = B_4Y$, $C_2 = ZC_1$, and $C_4 = WC_3$, (21) is reduced to

$$\begin{aligned}&\min_{V_1,V_2} r(E_1 \; S \; E_3) \\ &= \sum_{i=1}^{4} q_i + p_1 + p_2 + r(A \quad B_2 \quad B_4) + r\begin{pmatrix} A & B_4 \\ C_1 & O \end{pmatrix} \\ &+ \max\left\{ r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix} - r\begin{pmatrix} B_1 & A & B_4 \\ O & C_1 & O \end{pmatrix} - r\begin{pmatrix} C_2 & O & O \\ A & B_2 & B_4 \end{pmatrix}, \; -r\begin{pmatrix} A & B_2 & B_4 \\ C_1 & O & O \end{pmatrix} \right\} \\ &= \sum_{i=1}^{4} q_i + p_1 + p_2 + r(A \quad B_2 \quad B_4) + r\begin{pmatrix} A & B_4 \\ C_1 & O \end{pmatrix} + r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix} - r\begin{pmatrix} B_1 & A & B_4 \\ O & C_1 & O \end{pmatrix} \\ &- r\begin{pmatrix} C_2 & O & O \\ A & B_2 & B_4 \end{pmatrix}.\end{aligned} \tag{22}$$

The last equality holds, since the well-known Frobenius rank inequality $r(ABC) \geq r(AB) + r(BC) - r(B)$, then

$$\begin{aligned} r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix} &= r\begin{pmatrix} O & ZC_1 & O \\ B_2X & A & B_4 \end{pmatrix} \\ &= r\left(\begin{pmatrix} Z & O \\ O & I \end{pmatrix}\begin{pmatrix} O & C_1 & O \\ B_2 & A & B_4 \end{pmatrix}\begin{pmatrix} X & O & O \\ O & I & O \\ O & O & I \end{pmatrix}\right) \\ &\geq r\left(\begin{pmatrix} Z & O \\ O & I \end{pmatrix}\begin{pmatrix} O & C_1 & O \\ B_2 & A & B_4 \end{pmatrix}\right) + r\left(\begin{pmatrix} O & C_1 & O \\ B_2 & A & B_4 \end{pmatrix}\begin{pmatrix} X & O & O \\ O & I & O \\ O & O & I \end{pmatrix}\right) \\ &\quad - r\begin{pmatrix} O & C_1 & O \\ B_2 & A & B_4 \end{pmatrix} \\ &= r\begin{pmatrix} O & C_2 & O \\ B_2 & A & B_4 \end{pmatrix} + r\begin{pmatrix} O & C_1 & O \\ B_1 & A & B_4 \end{pmatrix} - r\begin{pmatrix} O & C_1 & O \\ B_2 & A & B_4 \end{pmatrix}. \end{aligned}$$

With the similar method, we also have

$$\begin{aligned} \min_{V_1, V_2} r\begin{pmatrix} E_2 \\ S \\ E_4 \end{pmatrix} = \sum_{i=1}^{4} p_i + q_1 + q_2 + r\begin{pmatrix} A \\ C_1 \\ C_3 \end{pmatrix} + r\begin{pmatrix} A & B_2 \\ C_3 & O \end{pmatrix} + r\begin{pmatrix} O & C_2 \\ B_1 & A \\ O & C_3 \end{pmatrix} \\ - r\begin{pmatrix} B_1 & A \\ O & C_1 \\ O & C_3 \end{pmatrix} - r\begin{pmatrix} C_2 & O \\ A & B_2 \\ C_3 & O \end{pmatrix}, \end{aligned} \tag{23}$$

$$\begin{aligned} \min_{V_1, V_2} r\begin{pmatrix} O & E_2 \\ E_1 & S \end{pmatrix} = p_1 + p_2 + p_4 + q_1 + q_2 + q_3 + r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \end{pmatrix} + r\begin{pmatrix} O & C_4 \\ B_3 & A \\ O & C_1 \end{pmatrix} \\ + r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_1 & A \end{pmatrix} - r\begin{pmatrix} O & O & C_4 \\ O & O & C_1 \\ B_3 & B_1 & A \end{pmatrix} - r\begin{pmatrix} O & O & C_4 \\ O & O & C_2 \\ B_3 & B_2 & A \end{pmatrix}, \end{aligned} \tag{24}$$

$$\begin{aligned} \min_{V_1, V_2} r\begin{pmatrix} S & E_3 \\ E_4 & O \end{pmatrix} = q_1 + q_2 + q_4 + p_1 + p_2 + p_3 + r\begin{pmatrix} A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix} + r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} \\ + r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix} - r\begin{pmatrix} B_1 & A & B_4 \\ O & C_1 & O \\ O & C_3 & O \end{pmatrix} - r\begin{pmatrix} C_2 & O & O \\ A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix}. \end{aligned} \tag{25}$$

On the other hand, by the formula (1) in Lemma 1.1, we have

$$\begin{aligned} \min_{V_1, V_2} r\begin{pmatrix} E_2 \\ S \end{pmatrix} = p_4 + p_1 + p_2 + q_1 + q_2 + \min\left\{ r\begin{pmatrix} O & O & C_2 \\ B_3 & B_1 & A \\ O & O & C_3 \end{pmatrix}, r\begin{pmatrix} O & C_4 & O \\ B_3 & A & B_2 \\ O & C_3 & O \end{pmatrix}, \right. \\ \left. r\begin{pmatrix} B_3 & A \\ O & C_1 \\ O & C_3 \end{pmatrix} \right\}. \end{aligned} \tag{26}$$

$$\max_{V_1,\,V_2} r\begin{pmatrix} S \\ E_4 \end{pmatrix} = p_3 + p_1 + p_2 + q_1 + q_2 + \min\left\{ r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix},\ r\begin{pmatrix} C_4 & O & O \\ A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix},\ r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} \right\}, \tag{27}$$

$$\max_{V_1,\,V_2} r(E_1 \quad S) = q_3 + p_1 + p_2 + q_1 + q_2 + \min\left\{ r\begin{pmatrix} O & C_4 & O \\ O & C_2 & O \\ B_1 & A & B_4 \end{pmatrix},\ r\begin{pmatrix} C_4 & O & O \\ A & B_2 & B_4 \end{pmatrix},\ r\begin{pmatrix} C_4 & O \\ A & B_4 \\ C_1 & O \end{pmatrix} \right\}, \tag{28}$$

$$\max_{V_1,\,V_2} r(S \quad E_3) = q_3 + p_1 + p_2 + q_1 + q_2 + \min\left\{ r\begin{pmatrix} O & C_2 & O \\ B_1 & A & B_4 \\ O & C_3 & O \end{pmatrix},\ r\begin{pmatrix} A & B_2 & B_4 \\ C_3 & O & O \end{pmatrix},\ r\begin{pmatrix} A & B_4 \\ C_1 & O \\ C_3 & O \end{pmatrix} \right\}. \tag{29}$$

Contrasting (18), (20) and (22)-(29) yields (17).

Corollary 2.6 Let $P(V_1, V_2, V_3, V_4)$ be given as (4) and let $R(B_1) \subseteq R(B_2)$, $R(B_3) \subseteq R(B_4)$, $R(B_1) \subseteq R(B_2), R(B_3) \subseteq R(B_4), R(C_2^*) \subseteq R(C_1^*), R(C_4^*) \subseteq R(C_3^*)$. Then the matrix equation $A = B_1V_1C_1 + B_2V_2C_2 + B_3V_3C_3 + B_4V_4C_4$ is consistent if and only if the right side of (17) is zero.

3 Some applications to generalized Schur complement and partial matrix

As direct applications of the results in Section 2, we determine in this section the maximal and minimal ranks of the generalized Schur complement A-$BM^{(1)}C$-$DN^{(1)}G$ and the partial matrix $(A \ \ BM^{(1)}C \ \ DN^{(1)}G)$ with respect to two variant matrices $M^{(1)} \lfloor M\{1\}$ and $N^{(1)} \lfloor N\{1\}$.

Theorem 3.1 Let $A \lfloor C^{m \times n}$, $B \lfloor C^{m \times p}$, $C \lfloor C^{q \times n}$, $D \lfloor C^{m \times s}$, $G \lfloor C^{t \times n}$, $M \lfloor C^{q \times p}$, and $N \lfloor C^{t \times s}$.

Then

$$\max_{M^{(1)} \in M\{1\}, N^{(1)} \in N\{1\}} r(A - BM^{(1)}C - DN^{(1)}G) = \min\{\widehat{T}_1,\ \widehat{T}_2,\ \widehat{T}_3\}, \tag{30}$$

where

$$\widehat{T}_1 = \min\left\{ r\begin{pmatrix} A & D & B \\ G & N & O \\ C & O & M \end{pmatrix} - r(M) - r(N),\ r\begin{pmatrix} A & D & B \\ G & N & O \end{pmatrix} - r(N),\ \ r\begin{pmatrix} A & D \\ G & N \\ C & O \end{pmatrix} - r(N) \right\},$$

$$\widehat{T}_2 = \min\left\{ \begin{pmatrix} A & B & D \\ C & M & O \end{pmatrix} - r(M),\ r(A \quad B \quad D),\ r\begin{pmatrix} A & D \\ C & O \end{pmatrix} \right\},$$

$$\widehat{T}_3 = \min\left\{ r\begin{pmatrix} A & B \\ C & M \\ G & O \end{pmatrix} - r(M),\ r\begin{pmatrix} A \\ C \\ G \end{pmatrix},\ r\begin{pmatrix} A & B \\ G & O \end{pmatrix} \right\}.$$

Proof. Applying Lemma 1.2, we have

$$M^{(1)} = M^{\dagger} + F_M W_1 + W_2 E_M \tag{31}$$

and

$$N^{(1)} = N^{\dagger} + F_N W_3 + W_4 E_N, \tag{32}$$

where W_i, i = 1,2,3,4 are arbitrary, $E_M = I_q - MM^{\dagger}$ and $F_M = I_p - M^{\dagger}M$. Substituting the Equation (31) and Equation (32) into the generalized Schur complement $A - BM^{(1)}C - DN^{(1)}G$ yields

$$A - BM^{(1)}C - DN^{(1)}G = A_1 - BF_M W_1 C - BW_2 E_M C - DF_N W_3 G - DW_4 E_N G, \tag{33}$$

where $A_1 = A - BM^{\dagger}C - DN^{\dagger}G$.

In fact $A_1 - BF_M W_1 C - BW_2 E_M C - DF_N W_3 G - DW_4 E_N G$ is a special case of the matrix expression $P(V_1,V_2,V_3,V_4)$, and $R(BF_M) \subseteq R(B)$, $R(DF_N) \subseteq R(D)$, $R((E_M C)^*) \subseteq R(C^*)$, $R((E_N G)^*) \subseteq R(G^*)$. In this case, from the formula (12) in Theorem 2.3, we have

$$\begin{aligned} &\max_{M^{(1)}\in M\{1\},\, N^{(1)}\in N\{1\}} r(A - BM^{(1)}C - DN^{(1)}G) \\ &= \max_{W_1, W_2, W_3, W_4} r(A_1 - BF_M W_1 C - BW_2 E_M C - DF_N W_3 G - DW_4 E_N G) \\ &= \min\{T_1', T_2', T_3'\}, \end{aligned} \tag{34}$$

where

$$T_1' = \min\left\{ r\begin{pmatrix} O & O & E_N G \\ O & O & E_M C \\ DF_N & BF_M & A_1 \end{pmatrix} r\begin{pmatrix} O & E_N G & O \\ DF_N & A_1 & B \end{pmatrix}, r\begin{pmatrix} O & E_N G \\ DF_N & A_1 \\ O & C \end{pmatrix} \right\}$$

$$T_2' = \min\left\{ r\begin{pmatrix} O & E_M C & O \\ BF_M & A_1 & D \end{pmatrix},\ r(A_1\ B\ B),\ r\begin{pmatrix} A_1 & D \\ C & O \end{pmatrix} \right\}$$

$$T_3' = \min\left\{ r\begin{pmatrix} O & E_M C \\ BF_M & A_1 \\ O & G \end{pmatrix},\ r\begin{pmatrix} A_1 & B \\ G & O \end{pmatrix},\ r\begin{pmatrix} A_1 \\ C \\ G \end{pmatrix} \right\}$$

For T_1', simplifying the ranks of matrices by Lemma 1.3 and block Gaussian elimination, we find that:

$$\begin{aligned} r\begin{pmatrix} O & O & E_N G \\ O & O & E_M C \\ DF_N & BF_M & A_1 \end{pmatrix} &= \begin{pmatrix} A_1 & DF_N & BF_M \\ E_M G & O & O \\ E_M C & O & O \end{pmatrix} \\ &= r\begin{pmatrix} A_1 & D & B & O & O & O \\ O & N & O & O & O & O \\ O & O & M & O & O & O \\ G & O & O & O & N & O \\ C & O & O & O & O & M \end{pmatrix} - 2r(N) - 2r(M) \\ &= r\begin{pmatrix} A & D & B \\ G & N & O \\ C & O & M \end{pmatrix} - r(N) - r(M), \end{aligned} \tag{35}$$

$$\begin{aligned} r\begin{pmatrix} O & E_N G & O \\ DF_N & A_1 & B \end{pmatrix} &= r\begin{pmatrix} A_1 & DF_N & B \\ E_N G & O & O \end{pmatrix} = r\begin{pmatrix} A_1 & D & B & O \\ O & N & O & O \\ G & O & O & N \end{pmatrix} - 2r(N) \\ &= r\begin{pmatrix} A & D & B \\ G & N & O \end{pmatrix} - r(N), \end{aligned} \tag{36}$$

$$r\begin{pmatrix} O & E_N G \\ DF_N & A_1 \\ O & C \end{pmatrix} = \begin{pmatrix} O & G & N & O \\ O & C & O & M \\ D & A_1 & O & O \\ N & O & O & O \\ O & C & O & O \end{pmatrix} - 2r(N) = r\begin{pmatrix} A & D \\ G & N \\ C & O \end{pmatrix} - r(N). \tag{37}$$

Combining the rank equalities (35), (36) with (37), we have $T_1' = \widehat{T_1}$.

By the similar approach, we also have $T_2' = \widehat{T_2}$ and $T_3' = \widehat{T_3}$. Then we have complete the proof of theorem.

Corollary 3.2 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$. Then the identity $A = BM^{(1)}C + DN^{(1)}G$ holds for every $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$ if and only if $\widehat{T_1} = O$ or $\widehat{T_2} = O$ or $\widehat{T_3} = O$.

From the proof of Theorem 3.1, we known that $A - BM^{(1)}C - DN^{(1)}G = A_1 - BF_mW_1C - BW_2E_mC - DF_NW_3G - DW_4E_NG$, where $A_1 = A - BM^{\dagger}C - DN^{\dagger}G$. In this case, $A - BM^{(1)}C - DN^{(1)}G$ is a special case of the matrix expression $P(V_1, V_2, V_3, V_4)$, and $R(BF_M) \subseteq R(B)$, $R(DF_N) \subseteq R(D)$, $R((E_MC)^*) \subseteq R(C^*)$, $R((E_NG)^*) \subseteq R(G^*)$. Then from the Theorem 2.5, we have

Theorem 3.3 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$.

Then

$$\begin{aligned} &\min_{M^{(1)} \in M\{1\},\, N^{(1)} \in N\{1\}} r(A - BM^{(1)}C - DN^{(1)}G) \\ &= \min_{W_1,\, W_2,\, W_3,\, W_4} r(A_1 - BF_MW_1C - BW_2E_MC - DF_NW_3G - DW_4E_NG) \\ &= r(A\ B\ D) + r\begin{pmatrix} A & D \\ C & O \end{pmatrix} + r\begin{pmatrix} M & C & O \\ B & A & D \end{pmatrix} + r\begin{pmatrix} A \\ C \\ G \end{pmatrix} + r\begin{pmatrix} A & B \\ G & O \end{pmatrix} + r\begin{pmatrix} M & C \\ B & A \\ O & G \end{pmatrix} \\ &-r\begin{pmatrix} B & A & D \\ M & O & O \\ O & C & O \end{pmatrix} - r\begin{pmatrix} O & C & O & M \\ B & A & D & O \end{pmatrix} - r\begin{pmatrix} B & A \\ M & O \\ O & C \\ O & G \end{pmatrix} - r\begin{pmatrix} C & O & M \\ A & B & O \\ G & O & O \end{pmatrix} + 3r(M) \\ &+\max\left\{ r\begin{pmatrix} N & G & O \\ D & A & B \end{pmatrix} + r\begin{pmatrix} O & G & N \\ B & A & O \\ O & C & O \end{pmatrix} + r\begin{pmatrix} N & O & G \\ O & M & C \\ D & B & A \end{pmatrix} - r\begin{pmatrix} N & O & G \\ O & O & C \\ D & B & A \\ O & M & O \end{pmatrix} \right. \\ &-r\begin{pmatrix} N & O & G & O \\ O & O & C & M \\ D & B & A & O \end{pmatrix} + r(N) - \delta_1 - \delta_2, \\ &r\begin{pmatrix} A & B & D \\ G & O & O \end{pmatrix} + r\begin{pmatrix} A & B \\ C & O \\ G & O \end{pmatrix} + r\begin{pmatrix} M & C & O \\ B & A & D \\ O & G & O \end{pmatrix} - r\begin{pmatrix} B & A & D \\ M & O & O \\ O & C & O \\ O & G & O \end{pmatrix} \\ &\left. -r\begin{pmatrix} C & O & O & M \\ A & B & D & O \\ G & O & O & O \end{pmatrix} - 2\delta_3 \right\}, \end{aligned} \tag{38}$$

where

$$\delta_1 = \min\left\{ r\begin{pmatrix} O & M & C \\ D & B & A \\ N & O & O \\ O & O & G \end{pmatrix} - r(M),\ r\begin{pmatrix} D & A & B \\ N & O & O \\ O & G & O \end{pmatrix},\ r\begin{pmatrix} D & A \\ N & O \\ O & C \\ O & G \end{pmatrix} \right\},$$

$$\delta_2 = \min\left\{ r\begin{pmatrix} O & G & O & N \\ M & C & O & O \\ B & A & D & O \end{pmatrix} - r(M),\ r\begin{pmatrix} G & O & O & N \\ A & B & D & O \end{pmatrix},\ r\begin{pmatrix} G & O & N \\ A & D & O \\ C & O & O \end{pmatrix} \right\},$$

$$\delta_3 = \min\left\{ r\begin{pmatrix} M & C & O \\ B & A & D \\ O & G & O \end{pmatrix} - r(M),\ r\begin{pmatrix} A & B & D \\ G & O & O \end{pmatrix},\ r\begin{pmatrix} A & D \\ C & O \\ G & O \end{pmatrix} \right\}.$$

Corollary 3.4 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$. then the identity $A = BM^{(1)}C + DN^{(1)}G$ is consistent if and only if the right side of (38) is zero.

Next, we will determine the maximal and minimal ranks of the partial matrix

$$(A \quad BM^{(1)}C \quad DN^{(1)}G)$$

with respect to $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$, by applying the results in Section 2.

It is quite obvious that the partial matrix $(A\ BM^{(1)}C\ DN^{(1)}G)$ may be written as

$$(A \quad BM^{(1)}C \quad DN^{(1)}G) = (A \quad O \quad O) + BM^{(1)}(O \quad C \quad O) + DN^{(1)}(O \quad O \quad G). \tag{39}$$

Then from (39) and Theorems 2.3 and 3.1, we have

Theorem 3.5 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$.

Then

$$\max_{M^{(1)} \in M\{1\}, N^{(1)} \in N\{1\}} r(A \quad BM^{(1)}C \quad DN^{(1)}G) = \min\left\{\tilde{T}_1,\ \tilde{T}_2,\ \tilde{T}_3\right\}, \tag{40}$$

where

$$\tilde{T}_1 = \min\left\{ r\begin{pmatrix} A & O & O & D & B \\ O & O & G & N & O \\ O & C & O & O & M \end{pmatrix} - r(M) - r(N),\ r\begin{pmatrix} A & O & D & B \\ O & G & N & O \end{pmatrix} - r(N),\right.$$
$$\left. r\begin{pmatrix} A & O & D \\ O & G & N \end{pmatrix} + r(C) - r(N) \right\},$$
$$\tilde{T}_2 = \min\left\{ r\begin{pmatrix} A & O & B & D \\ O & C & M & O \end{pmatrix} - r(M),\ r(A\ B\ D),\ r(A\ D) + r(C) \right\},$$
$$\tilde{T}_3 = \min\left\{ r\begin{pmatrix} A & O & B \\ O & C & M \end{pmatrix} + r(G) - r(M),\ r(A\ B) + r(G),\ r(A) + r(C) + r(G) \right\}.$$

Corollary 3.6 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$. then the inclusion $R(BM^{(1)}C + DN^{(1)}G) \subseteq R(A)$ holds for every $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$ if and only if $\tilde{T}_1 = O$ or $\tilde{T}_2 = O$ or $\tilde{T}_3 = O$.

On the other hand, from (39) and Theorems 2.5 and 3.3, we can easily obtain the minimal rank of the partial matrix $(A\ BM^{(1)}C\ DN^{(1)}G)$ with respect to $M^{(1)} \in M\{1\}$ and $N^{(1)} \in N\{1\}$.

Theorem 3.7 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \in C^{m \times s}$, $G \in C^{t \times n}$, $M \in C^{q \times p}$, and $N \in C^{t \times s}$.

Then

$$
\begin{aligned}
&\min_{M^{(1)} \in M\{1\}, N^{(1)} \in N\{1\}} r(A \quad BM^{(1)}C \quad DN^{(1)}G) \\
&= r(A) + r(A \quad D) + r(A \quad B) + r(A \quad B \quad D) + r\begin{pmatrix} M & O & C & O \\ B & A & O & D \end{pmatrix} \\
&-r\begin{pmatrix} B & A \\ M & O \end{pmatrix} - r\begin{pmatrix} B & A & D \\ M & O & O \end{pmatrix} - r\begin{pmatrix} O & O & C & O & M \\ B & A & O & D & O \end{pmatrix} + 3r(M) \\
&+ \max\left\{ r\begin{pmatrix} N & O & G & O \\ D & A & O & B \end{pmatrix} + r\begin{pmatrix} O & O & G & N \\ B & A & O & O \end{pmatrix} + r\begin{pmatrix} N & O & O & O & G \\ O & M & O & C & O \\ D & B & A & O & O \end{pmatrix} \right. \\
&-r\begin{pmatrix} N & O & O & G \\ D & B & A & O \\ O & M & O & O \end{pmatrix} - r\begin{pmatrix} N & O & O & O & G & Oa \\ O & O & O & C & O & M \\ D & B & A & O & O & O \end{pmatrix} + r(N) - \xi_1 - \xi_2, \\
&r(G) + r(A \; D) + r(A \; B \; D) + r\begin{pmatrix} M & O & C & O \\ B & A & O & D \\ O & G & O & \end{pmatrix} - r\begin{pmatrix} B & A & D \\ M & O & O \end{pmatrix} \\
&\left. -r\begin{pmatrix} O & C & O & O & M \\ A & O & B & D & O \end{pmatrix} - 2\xi_3 \right\},
\end{aligned}
\tag{41}
$$

where

$$
\begin{aligned}
\xi_1 &= \min\left\{ r\begin{pmatrix} O & M & O & C \\ D & B & A & O \\ N & O & O & O \end{pmatrix} + r(G) - r(M), \quad r\begin{pmatrix} D & A & B \\ N & O & O \end{pmatrix} + r(G), \right. \\
&\left. r\begin{pmatrix} D & A \\ N & O \end{pmatrix} + r(C) + r(G) \right\} \\
\xi_2 &= \min\{ r\begin{pmatrix} O & O & O & G & O & N \\ M & O & C & O & O & O \\ B & A & O & O & D & O \end{pmatrix} - r(M), \; r\begin{pmatrix} O & G & O & O & N \\ A & O & B & D & O \end{pmatrix}, \\
& r\begin{pmatrix} O & O & G & O & N \\ A & O & O & D & O \\ O & C & O & O & O \end{pmatrix}, \\
\xi_3 &= \min\{ r\begin{pmatrix} M & O & C & O \\ B & A & O & D \end{pmatrix} + r(G) - r(M), \; r(A \; B \; D) + r(G), \; r(A \; D) + r(C) + r(G)\}.
\end{aligned}
$$

Corollary 3.8 Let $A \in C^{m \times n}$, $B \in C^{m \times p}$, $C \in C^{q \times n}$, $D \lfloor C^{m \times s}$, $G \lfloor C^{t \times n}$, $M \lfloor C^{q \times p}$, and $N \lfloor C^{t \times s}$. then there are some $M^{(1)} \lfloor M\{1\}$ and $N^{(1)} \lfloor N\{1\}$, such that the inclusion $R(BM^{(1)}C+DN^{(1)}G) \subseteq R(A)$ holds if and only if the right side of (41) is zero.

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Authors' contributions

The authors jointly worked on deriving the results. All authors read and approved the final manuscript.

Competing interests

The authors declare that they have no competing interests.

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