Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

Fixed point theorems for a generalized almost (ϕ, φ)-contraction with respect to S in ordered metric spaces

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Abstract
In this paper, the existence theorems of fixed points and common fixed points for two weakly increasing mappings satisfying a new condition in ordered metric spaces are proved. Our results extend, generalize and unify most of the fundamental metrical fixed point theorems in the literature.
MSC: 47H09; 47H10

Keywords: coincidence points; common fixed points; generalized multi-valued f-weak contraction; generalized multi-valued (f, α, β)-weak contraction; T-weakly commuting

1 Introduction and preliminaries

The classical Banach contraction principle is one of the most useful results in nonlinear analysis. In a metric space, the full statement of the Banach contraction principle is given by the following theorem.

Theorem 1.1 *Let (X, d) be a complete metric space and $T : X \to X$. If T satisfies*

$$d(Tx, Ty) \leq kd(x, y) \tag{1.1}$$

for all $x, y \in X$, where $k \in [0, 1)$, then T has a unique fixed point.

Due to its simplicity and usefulness, it has become a very popular tool in solving existence problems in many branches of mathematical analysis and its has many applications in solving nonlinear equations. Then, several authors studied and extended it in many direction; for example, see [1–19] and the references therein.

Despite these important features, Theorem 1.1 suffers from one drawback: the contractive condition (1.1) forces T to be continuous on X. It was then natural to ask if there exist weaker contractive conditions which do not imply the continuity of T. In 1968, this question was answered in confirmation by Kannan [20], who extended Theorem 1.1 to mappings that need not be continuous on X (but are continuous at their fixed point, see [21]).

On the other hand, Sessa [22] introduced the notion of weakly commuting mappings, which are a generalization of commuting mappings, while Jungck [23] generalized the notion of weak commutativity by introducing compatible mappings and then weakly compatible mappings [24].

In 2004, Berinde [25] defined the notion of a weak contraction mapping which is more general than a contraction mapping. However, in [26] Berinde renamed it as an almost contraction mapping, which is more appropriate. Berinde [25] proved some fixed point theorems for almost contractions in complete metric spaces. Afterward, many authors have studied this problem and obtained significant results (see [27–36]). Moreover, in [25] Berinde proved that any strict contraction, the Kannan [26] and Zamfirescu [37] mappings as well as a large class of quasi-contractions are all almost contractions.

Let T and S be two self mappings in a metric space (X, d). The mapping T is said to be a S-contraction if there exists $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(Sx, Sy)$ for all $x, y \in X$.

In 2006, Al-Thagafi and Shahzad [38] proved the following theorem which is a generalization of many known results.

Theorem 1.2 ([38, Theorem 2.1]) *Let E be a subset of a metric space (X, d) and S, T be two selfmaps of E such that $T(E) \subseteq S(E)$. Suppose that S and T are weakly compatible, T is an S-contraction and $S(E)$ is complete. Then S and T have a unique common fixed point in E.*

Recently Babu *et al.* [39] defined the class of mappings satisfying condition (B) as follows.

Definition 1.3 Let (X, d) be a metric space. A mapping $T : X \to X$ is said to satisfy condition (B) if there exist a constant $\delta \in (0, 1)$ and some $L \geq 0$ such that

$$d(Tx, Ty) \leq \delta d(x, y) + L \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{1.2}$$

for all $x, y \in X$.

They proved a fixed point theorem for such mappings in complete metric spaces. They also discussed quasi-contraction, almost contraction and the class of mappings that satisfy condition (B) in detail.

In recent year, Ćirić *et al.* [40] defined the following class of mappings satisfying an almost generalized contractive condition.

Definition 1.4 Let (X, d) be a metric space, and let $S, T : X \to X$. A mapping T is called an almost generalized contraction if there exist $\delta \in [0, 1)$ and $L \geq 0$ such that

$$d(Tx, Sy) \leq \delta M(x, y) + L \min\{d(x, Tx), d(y, Sy), d(x, Sy), d(y, Tx)\} \tag{1.3}$$

for all $x, y \in X$, where $M(x, y) = \max\{d(x, y), d(x, Tx), d(y, Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\}$.

Definition 1.5 Let $(X, \leq)$ be a partial ordered set. We say that $x, y \in X$ are comparable if $x \leq y$ or $y \leq x$ holds.

Definition 1.6 Let $(X, \leq)$ be a partial ordered set. A mapping $T : X \to X$ is said to be nondecreasing if $Tx \leq Ty$, whenever $x, y \in X$ and $x \leq y$.

Definition 1.7 Let $(X, \leq)$ be a partial ordered set. Two mappings $S, T : X \to X$ are said to be strictly increasing if $Sx < TSx$ and $Tx < STx$ for all $x \in X$.

In 2004, Ran and Reurings [41] proved the following result.

Theorem 1.8 *Let $(X, \leq)$ be a partially ordered set such that every pair $\{x, y\} \subset X$ has a lower and an upper bound. Suppose that d is a complete metric on X. Let $T : X \to X$ be a continuous and monotone mapping. Suppose that there exists $\delta \in [0,1)$ such that $d(Tx, Ty) \leq \delta d(x,y)$ for all comparable $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a unique fixed point $p \in X$.*

Ćirić *et al.* in [40] established fixed point and common fixed point theorems which are more general than Theorem 1.8 and several comparable results in the existing literature regarding the existence of a fixed point in ordered spaces.

In this paper, we introduce a new class which extends and unifies mappings satisfying the almost generalized contractive condition and establish the result on the existence of fixed points and common fixed points in a complete ordered space. This result substantially generalizes, extends and unifies the main results of Ćirić *et al.* [40, Theorems 2.1, 2.2, 2.3, 2.6], Theorem 1.8, and several comparable results in the existing literature regarding the existence of a fixed and a common fixed point in ordered spaces.

2 Fixed point theorems for a generalized almost (ϕ, φ)-contraction

First we introduce the notion of generalized almost (ϕ, φ)-contraction mappings.

Definition 2.1 Let $(X, \leq)$ be a partially ordered set, and let a metric d exist on X. A mapping $T : X \to X$ is called a generalized almost (ϕ, φ)-contraction if there exist two mappings $\phi : X \to [0,1)$ which $\phi(Tx) \leq \phi(x)$ and $\varphi : X \to [0,\infty)$ such that

$$d(Tx, Ty) \leq \phi(x)M(x,y) + \varphi(x) \min\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\} \tag{2.1}$$

for all comparable $x, y \in X$, where $M(x,y) = \max\{d(x,y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$.

Theorem 2.2 *Let $(X, \leq)$ be a partially ordered set, and let a complete metric d exist on X. Let $T : X \to X$ be a strictly increasing continuous mapping with respect to $\leq$ and a generalized almost (ϕ, φ)-contraction mapping. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a unique fixed point in X.*

Proof If $Tx_0 = x_0$, then x_0 is fixed of T and we finish the proof. Now, we may assume that $Tx_0 \neq x_0$, that is, $x_0 < Tx_0$. We construct the sequence $\{x_n\}$ in X by

$$x_{n+1} = T^{n+1}x_0 = Tx_n \tag{2.2}$$

for all $n \geq 0$. Since T is strictly increasing, we have that $Tx_0 < T^2x_0 < \cdots < T^nx_0 < T^{n+1}x_0 < \cdots$. Thus, $x_1 < x_2 < \cdots < x_n < x_{n+1} < \cdots$, which implies that the sequence $\{x_n\}$ is strictly increasing. Note that

$$\begin{aligned} &M(x_n, x_{n+1}) \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\} \\ &\quad = \max\left\{d(x_n, x_{n+1}), d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\} \end{aligned}$$

$$\begin{aligned}
&= \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{d(x_n,x_{n+2})}{2}\right\} \\
&\le \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), \frac{d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2})}{2}\right\} \\
&= \max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\}.
\end{aligned} \tag{2.3}$$

Since x_n and x_{n+1} are comparable, we get

$$\begin{aligned}
d(x_{n+1},x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\
&\le \phi(x_n)M(x_n,x_{n+1}) \\
&\quad + \varphi(x_n)\min\left\{d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), d(x_n, Tx_{n+1}), d(x_{n+1}, Tx_n)\right\} \\
&= \phi(x_n)\max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\} \\
&\quad + \varphi(x_n)\min\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2}), d(x_n,x_{n+2}), d(x_{n+1},x_{n+1})\right\} \\
&= \phi(x_n)\max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\}.
\end{aligned} \tag{2.4}$$

If for some $n \in \mathbb{N}$, $\max\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\} = d(x_{n+1},x_{n+2})$, then we get

$$d(x_{n+1},x_{n+2}) \le \phi(x_n)d(x_{n+1},x_{n+2}) < d(x_{n+1},x_{n+2}), \tag{2.5}$$

which is a contradiction. Thus, we have

$$\max\left\{d(x_n,x_{n+1}), d(x_{n+1},x_{n+2})\right\} = d(x_n,x_{n+1}) \tag{2.6}$$

for all $n \in \mathbb{N}$. Therefore, from (2.4), we have

$$d(x_{n+1},x_{n+2}) \le \phi(x_n)d(x_n,x_{n+1})$$

for all $n \in \mathbb{N}$. Hence,

$$\begin{aligned}
d(x_{n+1},x_{n+2}) &\le \phi(x_n)d(x_n,x_{n+1}) \\
&\le \phi(x_n)\phi(x_{n-1})d(x_{n-1},x_n) \\
&\vdots \\
&\le \phi(x_n)\phi(x_{n-1})\cdots\phi(x_1)\phi(x_0)d(x_0,x_1) \\
&= \phi\left(T^n x_0\right)\phi\left(T^{n-1}x_0\right)\cdots\phi(Tx_0)\phi(x_0)d(x_0,x_1) \\
&\le \phi\left(T^{n-1} x_0\right)\phi\left(T^{n-2}x_0\right)\cdots\phi(x_0)\phi(x_0)d(x_0,x_1) \\
&\vdots \\
&\le \underbrace{\phi(x_0)\phi(x_0)\ldots\phi(x_0)\phi(x_0)}_{(n+1)\text{-term}}d(x_0,x_1) \\
&= \left(\phi(x_0)\right)^{n+1}d(x_0,x_1)
\end{aligned} \tag{2.7}$$

for all $n \geq 1$. Now, for positive integers m and n with $m > n$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m) \\ &\leq \big(\phi(x_0)\big)^n d(x_0, x_1) + \big(\phi(x_0)\big)^{n+1} d(x_0, x_1) + \cdots + \big(\phi(x_0)\big)^{m-1} d(x_0, x_1) \\ &= \Big(\big(\phi(x_0)\big)^n + \big(\phi(x_0)\big)^{n+1} + \cdots + \big(\phi(x_0)\big)^{m-1}\Big) d(x_0, x_1) \\ &\leq \left(\frac{(\phi(x_0))^n}{1 - \phi(x_0)}\right) d(x_0, x_1). \end{aligned} \tag{2.8}$$

Since $\phi(x_0) \in [0,1)$, if we take the limit as $m, n \to 0$, then $d(x_n, x_m) \to 0$, which implies that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$. It follows from the continuity of T that $T(T^n x_0) = T^{n+1} x_0 \to z$ implies that $Tz = z$. Therefore, z is a fixed point of T. □

Example 2.3 Let $X = [0,1]$, the partial order $\leq$ be defined by $a \leq b$ if and only if $b - a \in [0, \infty)$ and d be the usual metric on X. Let $T : X \to X$ be defined by $Tx = \frac{x^2}{4}$ for all $x \in X$. It can be easily checked that T is a generalized almost (ϕ, φ)-contraction mapping with $\phi(x) = \frac{x+1}{4}$ and $\varphi(x) = \ln x$. Moreover, T is strictly increasing on X, and there exists $0 \in X$ such that $0 \leq T0$. Therefore, T satisfies the conditions of Theorem 2.2 and thus T has a fixed point 0.

The following corollaries follow immediately from the Theorem 2.2 with $\phi(x) = \delta$ and $\varphi(x) = L$.

Corollary 2.4 ([40, Theorem 2.1]) *Let $(X, \leq)$ be a partially ordered set, and let a complete metric d exist on X. Let $T : X \to X$ be a strictly increasing continuous mapping with respect to $\leq$. Suppose that there exist $\delta \in [0,1)$ and $L \geq 0$ such that*

$$d(Tx, Ty) \leq \delta M(x,y) + L \min\big\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} \tag{2.9}$$

for all comparable $x, y \in X$, where $M(x,y) = \max\{d(x,y), d(x, Tx), d(y, Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\}$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point in X.

Corollary 2.5 *Let $(X, \leq)$ be a partially ordered set, and let a complete metric d exist on X. Let $T : X \to X$ be a strictly increasing continuous mapping with respect to $\leq$. Assume that there exist two mappings $\phi : X \to [0,1)$ with $\phi(Tx) \leq \phi(x)$ and $\varphi : X \to [0, \infty)$ such that*

$$d(Tx, Ty) \leq \phi(x) M(x,y) + \varphi(x) \min\big\{d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\big\} \tag{2.10}$$

for all comparable $x, y \in X$, where $M(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\}$. If there exists $x_0 \in X$ such that $x_0 \leq Tx_0$, then T has a fixed point in X.

Proof Since (2.10) is a special case of the generalized almost (ϕ, φ)-contraction, the result follows from Theorem 2.2. □

Corollary 2.6 *Let $(X, \leq)$ be a partially ordered set, and let a complete metric d exist on X. Let $T : X \to X$ be a strictly increasing continuous mapping with respect to $\leq$. Assume that* *there exist two mappings $\phi : X \to [0,1)$ with $\phi(Tx) \le \phi(x)$ and $\varphi : X \to [0,\infty)$ such that*

$$d(Tx,Ty) \le \phi(x)d(x,y) + \varphi(x)\min\bigl\{d(x,Tx),d(y,Ty),d(x,Ty),d(y,Tx)\bigr\} \tag{2.11}$$

for all comparable $x, y \in X$. If there exists $x_0 \in X$ such that $x_0 \le Tx_0$, then T has a fixed point in X.

Proof Since (2.11) is a special case of the generalized almost (ϕ,φ)-contraction, the result follows from Theorem 2.2. □

Theorem 2.7 *Let $(X,\le)$ be a partially ordered set, and let a complete metric d exist on X. Let $T : X \to X$ be a strictly increasing mapping with respect to $\le$ and a generalized almost (ϕ,φ)-contraction mapping with a continuous mapping ϕ. If there exists $x_0 \in X$ such that $x_0 \le Tx_0$ and for an increasing sequence $\{x_n\}$ in X converging to $x \in X$ we have $\lim_{n\to\infty} \varphi(x_n) < \infty$ and $x_n \le x$ for all $n \in \mathbb{N}$, then T has a fixed point in X.*

Proof Suppose that $Tx \neq x$ for all $x \in X$. Following similar arguments to those given in Theorem 2.2, we obtain an increasing sequence $\{x_n\}$ such that $x_n \to z$ as $n \to \infty$ for some $z \in X$. By the given hypothesis, we have $x_n \le z$ for all $n \in \mathbb{N}$. Therefore,

$$\begin{aligned} d(x_{n+1},Tz) &= d(Tx_n,Tz) \\ &\le \phi(x_n)\max\left\{d(x_n,z),d(x_n,Tx_n),d(z,Tz),\frac{d(x_n,Tz)+d(z,Tx_n)}{2}\right\} \\ &\quad + \varphi(x_n)\min\bigl\{d(x_n,Tx_n),d(z,Tz),d(x_n,Tz),d(z,Tx_n)\bigr\} \\ &= \phi(x_n)\max\left\{d(x_n,z),d(x_n,x_{n+1}),d(z,Tz),\frac{d(x_n,Tz)+d(z,x_{n+1})}{2}\right\} \\ &\quad + \varphi(x_n)\min\bigl\{d(x_n,x_{n+1}),d(z,Tz),d(x_n,Tz),d(z,x_{n+1})\bigr\}. \end{aligned} \tag{2.12}$$

Taking the limit as $n \to \infty$, we get that $d(z,Tz) \le \phi(z)d(z,Tz)$. Since $\phi(z) \in [0,1)$, we have $d(z,Tz) = 0$, that is, $Tz = z$, which is a contradiction. Therefore, there exists $z_1 \in X$ such that $Tz_1 = z_1$ which implies that z_1 is a fixed point of T. □

3 Common fixed point theorems for a generalized almost (ϕ,φ)-contraction with respect to S

Definition 3.1 Let $(X,\le)$ be a partially ordered set, and let a metric d exist on X, and $S,T : X \to X$. A mapping T is called a generalized almost (ϕ,φ)-contraction with respect to S if there exist two mappings $\phi : X \to [0,1)$ with $\phi(Sx) \le \phi(x)$ and $\phi(Tx) \le \phi(x)$ and $\varphi : X \to [0,\infty)$ such that

$$d(Tx,Sy) \le \phi(x)M(x,y) + \varphi(x)\min\bigl\{d(x,Tx),d(y,Sy),d(x,Sy),d(y,Tx)\bigr\} \tag{3.1}$$

for all comparable $x, y \in X$, where $M(x,y) = \max\{d(x,y),d(x,Tx),d(y,Sy),\frac{d(x,Sy)+d(y,Tx)}{2}\}$.

Remark 3.2 If we take $\phi(x) = \delta$ where $\delta \in [0,1)$ and $\varphi(x) = L$ where $L \in [0,\infty)$, then the generalized almost (ϕ,φ)-contraction with respect to S reduces to an almost generalized contraction of Ćirić *et al.* in [40].

The following theorem deals with the existence of a common fixed point of two weakly increasing mappings which is more general and covers more than the result of Ćirić *et al.* in [40].

Theorem 3.3 *Let $(X, \preceq)$ be a partially ordered set, and let a complete metric d exist on X. Let $S, T : X \to X$ be two strictly weakly increasing mappings with respect to $\preceq$ and T be a generalized almost (ϕ, φ)-contraction with respect to S. If either S or T is continuous, then there exists a common fixed point of S and T in X.*

Proof Let x_0 be an arbitrary point in X. We construct the sequence $\{x_n\}$ in X such that

$$x_{2n+1} = Tx_{2n} \quad \text{and} \quad x_{2n+2} = Sx_{2n+1} \tag{3.2}$$

for all $n \geq 0$. Since S and T are strictly weakly increasing, we get

$$x_{2n+1} = Tx_{2n} \prec STx_{2n} = Sx_{2n+1} = x_{2n+2} \tag{3.3}$$

and

$$x_{2n+2} = Sx_{2n+1} \prec TSx_{2n+1} = Tx_{2n+2} = x_{2n+3} \tag{3.4}$$

for all $n \geq 0$. Therefore, $x_1 \prec x_2 \prec \cdots \prec x_n \prec x_{n+1} \prec \cdots$, that is, the sequence $\{x_n\}$ is strictly increasing. Note that

$$\begin{aligned}
&M(x_{2n}, x_{2n+1}) \\
&\quad = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Tx_{2n+1}), \frac{d(x_{2n}, Tx_{2n+1}) + d(x_{2n+1}, Tx_{2n})}{2}\right\} \\
&\quad = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1})}{2}\right\} \\
&\quad = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+2})}{2}\right\} \\
&\quad \leq \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2})}{2}\right\} \\
&\quad = \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right\}
\end{aligned} \tag{3.5}$$

for all $n \geq 0$. Since x_{2n} and x_{2n+1} are comparable, we get

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}) \\
&\quad = d(Tx_{2n}, Sx_{2n+1}) \\
&\quad \leq \phi(x_{2n})M(x_{2n}, x_{2n+1}) \\
&\qquad + \varphi(x_{2n}) \min\left\{d(x_{2n}, Tx_{2n}), d(x_{2n+1}, Sx_{2n+1}), d(x_{2n}, Sx_{2n+1}), d(x_{2n+1}, Tx_{2n})\right\} \\
&\quad = \phi(x_{2n}) \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right\} \\
&\qquad + \varphi(x_{2n}) \min\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), d(x_{2n}, x_{2n+2}), 0\right\} \\
&\quad = \phi(x_{2n}) \max\left\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\right\}.
\end{aligned} \tag{3.6}$$

If for some $n \in \mathbb{N}$, $\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n+1}, x_{2n+2})$, then we get

$$d(x_{2n+1}, x_{2n+2}) \leq \phi(x_{2n}) d(x_{2n+1}, x_{2n+2}) < d(x_{2n+1}, x_{2n+2}), \tag{3.7}$$

which is a contradiction. Thus,

$$\max\{d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2})\} = d(x_{2n}, x_{2n+1}) \tag{3.8}$$

for all $n \in \mathbb{N}$. Therefore, from (3.6), we have

$$d(x_{2n+1}, x_{2n+2}) \leq \phi(x_{2n}) d(x_{2n}, x_{2n+1})$$

for all $n \in \mathbb{N}$. Similarly, it can be proved that

$$d(x_{2n+3}, x_{2n+2}) \leq \phi(x_{2n+2}) d(x_{2n+2}, x_{2n+1})$$

for all $n \in \mathbb{N}$. It follows from the proof of Theorem 2.2 that

$$d(x_{n+1}, x_{n+2}) \leq (\phi(x_0))^{n+1} d(x_0, x_1) \tag{3.9}$$

for all $n \geq 1$. Similarly to the proof of Theorem 2.2, we get $\{x_n\}$ is a Cauchy sequence. It follows from the completeness of X that there exists a $z \in X$ such that $x_n \to z$ as $n \to \infty$. Suppose that T is continuous, then $T(T^n x_0) = T^{n+1} x_0 \to z$. This implies that $Tz = z$. Therefore, z is a fixed point of T. Suppose that $Sz \neq z$. Since $z \preceq z$. Therefore,

$$\begin{aligned} d(z, Sz) &= d(Tz, Sz) \\ &\leq \phi(z) \max\left\{d(z,z), d(z,Tz), d(z,Sz), \frac{d(z,Sz) + d(z,Tz)}{2}\right\} \\ &\quad + \varphi(z) \min\{d(z,Tz), d(z,Sz), d(z,Sz), d(z,Tz)\} \\ &\leq \phi(z) d(z, Sz), \end{aligned} \tag{3.10}$$

which is a contradiction. Therefore, $Sz = z$ and then z is a common fixed point of S and T. Similarly, it can be proved that S and T have a common fixed point if S is continuous. □

The following corollaries are a generalization and extension of Corollaries 2.4 and 2.5 in Ćirić *et al.* in [40].

Corollary 3.4 *Let $(X, \preceq)$ be a partially ordered set, and let a complete metric d exist on X. Let $S, T : X \to X$ be two strictly weakly increasing mappings with respect to $\preceq$. Assume that there exist two mappings $\phi : X \to [0,1)$ with $\phi(Sx) \leq \phi(x)$ and $\phi(Tx) \leq \phi(x)$ and $\varphi : X \to [0,\infty)$ such that*

$$d(Tx, Sy) \leq \phi(x) M(x,y) + \varphi(x) \min\{d(x,Tx), d(y,Sy), d(x,Sy), d(y,Tx)\} \tag{3.11}$$

for all comparable $x, y \in X$, where $M(x,y) = \max\{d(x,y), \frac{d(x,Tx)+d(y,Sy)}{2}, \frac{d(x,Sy)+d(y,Tx)}{2}\}$. If either S or T is continuous, then there exists a common fixed point of S and T in X.

Proof Since (3.11) is a special case of the generalized almost (ϕ,φ)-contraction with respect to S, the result follows from Theorem 3.3. □

Corollary 3.5 *Let $(X,\le)$ be a partially ordered set, and let a complete metric d exist on X. Let $S,T:X\to X$ be two strictly increasing mappings with respect to $\le$. Assume that there exist two mappings $\phi:X\to[0,1)$ with $\phi(Sx)\le\phi(x)$ and $\phi(Tx)\le\phi(x)$ and $\varphi:X\to[0,\infty)$ such that*

$$d(Tx,Sy)\le\phi(x)d(x,y)+\varphi(x)\min\bigl\{d(x,Tx),d(y,Sy),d(x,Sy),d(y,Tx)\bigr\} \tag{3.12}$$

for all comparable $x,y\in X$. If either S or T is continuous, then there exists a common fixed point of S and T in X.

Proof Since (3.12) is a special case of the generalized almost (ϕ,φ)-contraction with respect to S, the result follows from Theorem 3.3. □

Now, we have the following result of the continuity on the set of common fixed points. Let $F(S,T)$ denote the set of all common fixed points of S and T.

Theorem 3.6 *Let $(X,\le)$ be a partially ordered set, and let a complete metric d exist on X. Let $S,T:X\to X$ and T be a generalized almost (ϕ,φ)-contraction mapping with respect to S. If $F(S,T)\ne\emptyset$ and for any sequence $\{x_n\}$ in X with $x_n\to x$ as $n\to\infty$ for some $x\in X$, we have $x_n\le x$ for all $n\in\mathbb{N}$, then S and T are continuous at $z\in F(S,T)$.*

Proof Fix $z\in F(S,T)$. Let $\{x_n\}$ be any sequence in X converging to z and then $x_n\le z$ for all $n\in\mathbb{N}$. From the notion of a generalized almost (ϕ,φ)-contraction with respect to S, we get

$$\begin{aligned} d(Tz,Sx_n) &\le \phi(z)\max\left\{d(z,x_n),d(z,Tz),d(x_n,Sx_n),\frac{d(z,Sx_n)+d(x_n,Tz)}{2}\right\}\\ &\quad+\varphi(z)\min\bigl\{d(z,Tz),d(x_n,Sx_n),d(z,Sx_n),d(x_n,Tz)\bigr\}\\ &=\phi(z)\max\left\{d(z,x_n),0,d(x_n,Sx_n),\frac{d(z,Sx_n)+d(x_n,z)}{2}\right\}\\ &\quad+\varphi(z)\min\bigl\{0,d(x_n,Sx_n),d(z,Sx_n),d(x_n,Tz)\bigr\}\\ &=\phi(z)\max\left\{d(z,x_n),d(x_n,Sx_n),\frac{d(z,Sx_n)+d(x_n,z)}{2}\right\} \end{aligned} \tag{3.13}$$

for all $n\ge1$. Letting $n\to\infty$, we have $Sx_n\to z=Sz$. Therefore, S is continuous at $z\in F(S,T)$. Similarly, it can be shown that T is continuous at $z\in F(S,T)$. □

Remark 3.7 In fixed point theory, after the remarkable paper of Huang and Zhang [42], cone metric spaces have been considered by several authors. The results and theorems in this paper can be also generalized to cone metric spaces.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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Acknowledgements
This work was supported by the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission. The first author would like to thank the Research Professional Development Project Under the Science Achievement Scholarship of Thailand (SAST) for financial support during the preparation of this manuscript. The second author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education, Science and Technology (2012R1A1A2042138). This project was partially completed during the first and third authors' last visit to Kyungnam University.

Received: 27 July 2012 Accepted: 1 November 2012 Published: 13 November 2012

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doi:10.1186/1029-242X-2012-263
Cite this article as: Sintunavarat et al.: Fixed point theorems for a generalized almost (ϕ,φ)-contraction with respect to S in ordered metric spaces. *Journal of Inequalities and Applications* 2012 **2012**:263.