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A new version of Jensen's inequality and related results

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Abstract
In this paper we expand Jensen's inequality to two-variable convex functions and find the lower bound of the Hermite-Hadamard inequality for a convex function on the bounded area from the plane.

1 Introduction

Let μ be a positive measure on X such that $\mu(X) = 1$. If f is a real-valued function in $L^1(\mu)$, $a < f(x) < b$ for all $x \in X$ and φ is convex on (a, b), then

$$\varphi\left(\int_X f\,d\mu\right) \le \int_X (\varphi \circ f)\,d\mu. \tag{1}$$

The inequality (1) is known as Jensen's inequality [1].

In recent years, there have been many extensions, refinements and similar results of the inequality (1). Recall that the function $F : \Delta = [a, b] \times [c, d] \to \mathbb{R}$ is convex on Δ if

$$F\left(\lambda x + (1 - \lambda)z, \lambda y + (1 - \lambda)w\right) \le \lambda F(x, y) + (1 - \lambda)F(z, w)$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$. A function $F : \Delta \to \mathbb{R}$ is called co-ordinated convex on Δ if the partial functions $F_y : [a, b] \to \mathbb{R}$, $F_y(u) = F(u, y)$ and $F_x : [c, d] \to \mathbb{R}$, $F_x(v) = F(x, v)$ are convex for all $x \in [a, b]$ and $y \in [c, d]$. Note that every convex function $F : \Delta \to \mathbb{R}$ is co-ordinated convex, but the converse is not generally true; see [2]. Also note that if F is a convex function on $\mathbb{R}^2$ and g, h are real-valued functions such that $D_g = D_h = \mathbb{R}$, then $f(t) = F(g(t), h(t))$ may be not convex on $\mathbb{R}$.

In this paper under suitable conditions, we expand Jensen's inequality to two-variable convex functions and deduce some further important inequalities. Finally, we find a lower bound for the integral

$$\frac{1}{\int_a^b (g(x) - h(x))\,dx} \int_a^b \int_{h(x)}^{g(x)} F(x, y)\,dy\,dx,$$

where F is convex on the convex bounded area by $y = g(x)$, $y = h(x)$ and $x = a$, $x = b$.

2 Main results

Theorem 1 *Let p be a non-negative continuous function on* $[a,b]$ *such that* $\int_a^b p(x)\,dx > 0$. *If g and h are real-valued continuous functions on* $[a,b]$ *and*

$$m_1 \le g(x) \le M_1, \qquad m_2 \le h(x) \le M_2$$

for all $x \in [a,b]$, *and F is convex on*

$$\Delta = [m_1, M_1] \times [m_2, M_2],$$

then

$$F\left(\frac{\int_a^b g(t)p(t)\,dt}{\int_a^b p(t)\,dt}, \frac{\int_a^b h(t)p(t)\,dt}{\int_a^b p(t)\,dt}\right) \le \frac{\int_a^b F(g(t),h(t))p(t)\,dt}{\int_a^b p(t)\,dt} \tag{2}$$

and

$$F\left(\frac{\int_a^b g(t)\,dt}{b-a}, \frac{\int_a^b h(t)\,dt}{b-a}\right) \le \frac{1}{b-a}\int_a^b F\bigl(g(t),h(t)\bigr)\,dt. \tag{3}$$

The inequalities hold in reversed order if f is concave on Δ.

Proof Denote

$$\alpha(x) = \frac{\int_a^x g(t)p(t)\,dt}{\int_a^x p(t)\,dt}$$

and

$$\beta(x) = \frac{\int_a^x h(t)p(t)\,dt}{\int_a^x p(t)\,dt}.$$

Then by L'Hospital's rule, we have $\lim_{x\to a}\alpha(x) = g(a)$ and $\lim_{x\to a}\beta(x) = h(a)$. So, α and β are continuous on $[a,b]$. Denote

$$H(x) = F\bigl(\alpha(x),\beta(x)\bigr) - \frac{\int_a^x F(g(t),h(t))p(t)\,dt}{\int_a^x p(t)\,dt}.$$

We will show that $H(b) \le 0$. We have

$$\begin{aligned} H'(x) &= \frac{\partial F(\alpha(x),\beta(x))}{\partial\alpha}\alpha'(x) + \frac{\partial F(\alpha(x),\beta(x))}{\partial\beta}\beta'(x) \\ &\quad - \frac{F(g(x),h(x))p(x)}{\int_a^x p(t)\,dt} + p(x)\frac{\int_a^x F(g(t),h(t))p(t)\,dt}{(\int_a^x p(t)\,dt)^2}. \end{aligned}$$

By the convexity of F, we obtain

$$\begin{aligned} F\bigl(g(x),h(x)\bigr) - F\bigl(\alpha(x),\beta(x)\bigr) &\ge \frac{\partial F(\alpha(x),\beta(x))}{\partial\alpha}\bigl(g(x)-\alpha(x)\bigr) \\ &\quad + \frac{\partial F(\alpha(x),\beta(x))}{\partial\beta}\bigl(h(x)-\beta(x)\bigr). \end{aligned}$$

So, we get

$$
\begin{aligned}
H'(x) \le{}& \frac{\partial(\alpha(x),\beta(x))}{\partial\alpha}\alpha'(x) + \frac{\partial(\alpha(x),\beta(x))}{\partial\beta}\beta'(x) \\
&- \frac{p(x)}{\int_a^x p(t)\,dt}\biggl[F\bigl(\alpha(x),\beta(x)\bigr) + \frac{\partial(\alpha(x),\beta(x))}{\partial\alpha}\bigl(g(x)-\alpha(x)\bigr) \\
&+ \frac{\partial F(\alpha(x),\beta(x))}{\partial\beta}\bigl(h(x)-\beta(x)\bigr)\biggr] + p(x)\frac{\int_a^x F(g(t),h(t))p(t)\,dt}{(\int_a^x p(t)\,dt)^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
H'(x) \le{}& \frac{\partial(\alpha(x),\beta(x))}{\partial\alpha}\biggl[\alpha'(x) - \frac{p(x)}{\int_a^x p(t)\,dt}\bigl(g(x)-\alpha(x)\bigr)\biggr] \\
&+ \frac{\partial(\alpha(x),\beta(x))}{\partial\beta}\biggl[\beta'(x) - \frac{p(x)}{\int_a^x p(t)\,dt}\bigl(h(x)-\beta(x)\bigr)\biggr] \\
&- \frac{p(x)F(\alpha(x),\beta(x))}{\int_a^x p(t)\,dt} + p(x)\frac{\int_a^x F(g(t),h(t))g(t)\,dt}{(\int_a^x p(t)\,dt)^2}.
\end{aligned}
$$

By easy calculation, we see that

$$
\alpha'(x) - \frac{p(x)}{\int_a^x p(t)\,dt}\bigl(g(x)-\alpha(x)\bigr) = \beta'(x) - \frac{p(x)}{\int_a^x p(t)\,dt}\bigl(h(x)-\beta(x)\bigr) = 0.
$$

Therefore,

$$
H'(x) \le -\frac{p(x)}{\int_a^x p(t)\,dt}\biggl[F\bigl(\alpha(x),\beta(x)\bigr) - \frac{\int_a^x F(g(t),h(t))p(t)\,dt}{\int_a^x p(t)\,dt}\biggr] = -\frac{p(x)}{\int_a^x P(t)\,dt}H(x).
$$

Thus,

$$
\biggl(\int_a^x p(t)\,dt\biggr)H'(x) + p(x)H(x) \le 0 \quad\Rightarrow\quad \biggl[\biggl(\int_a^x p(t)\,dt\biggr)H(x)\biggr]' \le 0.
$$

So,

$$
\biggl(\int_a^b p(t)\,dt\biggr)H(b) \le 0 \quad\Rightarrow\quad H(b) \le 0.
$$

The proof is complete. For the proof of (3), set $p(x) = 1$.

Note the inequalities (2) and (3) are sharp because $F(x,y) = 1$. □

Corollary 1 *Let g and h be real-valued continuous functions. Then we have*

(i) *for* $\frac{1}{p} + \frac{1}{q} = 1$, $p,q > 1$,

$$
\int_a^b \bigl|g(t)\bigr|\bigl|h(t)\bigr|\,dt \le \biggl(\int_a^b \bigl|g(t)\bigr|^p\,dt\biggr)^{\frac{1}{p}}\biggl(\int_a^b \bigl|h(t)\bigr|^q\,dt\biggr)^{\frac{1}{q}} \quad \textit{Holder's inequality,}
$$

(ii) *for* $p \geq 1$,

$$\left(\int_a^b \left|g(t)+h(t)\right|^{\frac{1}{p}}\,dt\right)^p \geq \left(\int_a^b \left|g(t)\right|^{\frac{1}{p}}\,dt + \int_a^b \left|h(t)\right|^{\frac{1}{p}}\,dt\right)^p \quad \textit{reverse Minkowski's inequality},$$

(iii) *for* $p \geq 1$,

$$\left(\int_a^b \left|g(t)+h(t)\right|^p\,dt\right)^{\frac{1}{p}} \leq \left(\int_a^b \left|g(t)\right|^p\,dt\right)^{\frac{1}{p}} + \left(\int_a^b \left|h(t)\right|^p\,dt\right)^{\frac{1}{p}} \quad \textit{Minkowski's inequality},$$

(iv)

$$\ln\left(e^{\frac{1}{b-a}\int_a^b g(t)\,dt} + e^{\frac{1}{b-a}\int_a^b h(t)\,dt}\right) \leq \frac{1}{b-a}\int_a^b \ln\left(e^{g(t)}+e^{h(t)}\right)dt.$$

Proof

(i) The function

$$F(x,y) = |x|^{\frac{1}{p}}|y|^{\frac{1}{q}} \quad \left(\frac{1}{p}+\frac{1}{q}=1\right),$$

is concave, so by the inequality (3), we have

$$\frac{(\int_a^b |g(t)|\,dt)^{\frac{1}{p}}}{(b-a)^{\frac{1}{p}}} \times \frac{(\int_a^b |h(t)|\,dt)^{\frac{1}{q}}}{(b-a)^{\frac{1}{q}}} \geq \frac{\int_a^b |g(t)|^{\frac{1}{p}}|h(t)|^{\frac{1}{q}}\,dt}{b-a}.$$

Hence,

$$\int_a^b \left|g(t)\right|^{\frac{1}{p}}\left|h(t)\right|^{\frac{1}{q}}\,dt \leq \left(\int_a^b \left|g(t)\right|dt\right)^{\frac{1}{p}}\left(\int_a^b \left|h(t)\right|dt\right)^{\frac{1}{q}}.$$

Now, set $|g(t)| \to |g(t)|^p$ and $|h(t)| \to |h(t)|^q$. We obtain

$$\int_a^b \left|g(t)\right|\left|h(t)\right|dt \leq \left(\int_a^b \left|g(t)\right|^p\,dt\right)^{\frac{1}{p}} + \left(\int_a^b \left|h(t)\right|^q\,dt\right)^{\frac{1}{q}}.$$

(ii) The function $F(x,y) = (|x|^p+|y|^p)^{\frac{1}{p}}$ is convex for $p \geq 1$ and is concave for $p < 1$. So, by the inequality (3), we have

$$\left[\frac{(\int_a^b |g(t)|\,dt)^p}{(b-a)^p} + \frac{(\int_a^b |h(t)|\,dt)^p}{(b-a)^p}\right]^{\frac{1}{p}} \leq \frac{\int_a^b (|g(t)|^p+|h(t)|^p)^{\frac{1}{p}}\,dt}{b-a}$$

so

$$\int_a^b \left(\left|g(t)\right|^p+\left|h(t)\right|^p\right)^{\frac{1}{p}}\,dt \geq \left[\left(\int_a^b \left|g(t)\right|dt\right)^p + \left(\int_a^b \left|h(t)\right|dt\right)^p\right]^{\frac{1}{p}}.$$

Now, set $|g(t)| \to |g(t)|^{\frac{1}{p}}$ and $|h(t)| \to |h(t)|^{\frac{1}{p}}$. We get

$$\int_a^b \left(|g(t)| + |h(t)|\right)^{\frac{1}{p}} dt \geq \left[\left(\int_a^b |g(t)|^{\frac{1}{p}} dt\right)^p + \left(\int_a^b |h(t)|^{\frac{1}{p}} dt\right)^p\right]^{\frac{1}{p}}.$$

So,

$$\left(\int_a^b \left(|g(t)| + |h(t)|\right)^{\frac{1}{p}} dt\right)^p \geq \left(\int_a^b |g(t)|^{\frac{1}{p}} dt\right)^p + \left(\int_a^b |h(t)|^{\frac{1}{p}} dt\right)^p.$$

The proof of (iii) is similar to that of (ii) and can be omitted. For the proof of (iv), note $f(x,y) = \ln(e^x + e^y)$ is convex on $\mathbb{R}^2$. Now, apply the inequality (3). □

Remark 1 By similar assumptions, we can prove Theorem 1 for an n-variable convex function F on $\mathbb{R}^n$ and obtain the inequality

$$F\left(\frac{\int_a^b g_1(t)\,dt}{b-a}, \ldots, \frac{\int_a^b g_n(t)\,dt}{b-a}\right) \leq \frac{1}{b-a}\int_a^b F\left(g_1(t), \ldots, g_n(t)\right) dt.$$

In particular, we can obtain a similar inequality for Holder and Minkowski inequalities. For example, by the concavity of

$$F(t_1, t_2, \ldots, t_n) = \prod_{i=1}^n |t_i|^{\frac{1}{p_i}} \quad \left(\sum_{i=1}^n \frac{1}{pi} = 1\right),$$

we can get the inequality

$$\int_a^b \left(\prod_{i=1}^n |g_i|\right) dt \leq \prod_{i=1}^n \left(\int_a^b |g_i|^{p_i}\right)^{\frac{1}{p_i}}.$$

3 Hermite-Hadamard inequality

Let $f : [a,b] \to \mathbb{R}$ be a convex function, then the following inequality is known as the Hermite-Hadamard inequality [3] and [4]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{4}$$

In [5], Dragomir established the following similar inequality (4) for convex functions on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

Theorem 2 *Suppose* $f : \triangle = [a,b] \times [c,d] \to \mathbb{R}$ *is a convex function on the co-ordinates on* $\triangle$. *Then one has the inequalities*

$$\begin{aligned} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x,y)\,dy\,dx \\ &\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Also Dragomir investigated the Hermite-Hadamard inequality on the disk [6] and [7].

In [8], Matejíčka proved the left-hand side of the Hermite-Hadamard inequality of several variables for a convex function on certain convex compact sets. In the following theorem, we prove the left-hand side of the Hermite-Hadamard inequality in another way and as a result of Theorem 2.

Theorem 3 *Let $\triangle$ be a bounded area by a convex function h and a concave function g on $[a,b]$ such that for any $x \in [a,b]$, $g(x) \geq h(x)$. Also, let F be a two-variable convex function on $\triangle$. Then one has the inequality*

$$F\left(\frac{\int_a^b x(g(x)-h(x))\,dx}{\int_a^b (g(x)-h(x))\,dx}, \frac{\frac{1}{2}\int_a^b (g^2(x)-h^2(x))\,dx}{\int_a^b (g(x)-h(x))\,dx}\right) \leq \frac{\int_a^b \int_{h(x)}^{g(x)} F(x,y)\,dy\,dx}{\int_a^b (g(x)-h(x))\,dx}.$$

Proof Since F is convex on $\triangle$, hence f is co-ordinated convex on $\triangle$. So, $F_x : [h(x), g(x)] \to \mathbb{R}$, $F_x(y) = F(x,y)$ is convex on $[h(x), g(x)]$ for all $x \in [a,b]$. By the left-hand side of the Hermite-Hadamard inequality (4), we have

$$\bigl(g(x)-h(x)\bigr)F\left(x, \frac{g(x)+h(x)}{2}\right) \leq \int_{h(x)}^{g(x)} F(x,y)\,dy.$$

Integrating this inequality on $[a,b]$, we obtain

$$\int_a^b \bigl(g(x)-h(x)\bigr)F\left(x, \frac{g(x)+h(x)}{2}\right)dx \leq \int_a^b \int_{h(x)}^{g(x)} F(x,y)\,dy\,dx.$$

So,

$$\frac{\int_a^b (g(x)-h(x))F(x, \frac{g(x)+h(x)}{2})\,dx}{\int_a^b (g(x)-h(x))\,dx} \leq \frac{1}{\int_a^b (g(x)-h(x))\,dx}\int_a^b \int_{h(x)}^{g(x)} F(x,y)\,dy\,dx.$$

Now, let $p(x) = g(x) - h(x)$. By the inequality (2), we get

$$\begin{aligned} F\left(\frac{\int_a^b x(g(x)-h(x))\,dx}{\int_a^b (g(x)-h(x))\,dx}, \frac{\frac{1}{2}\int_a^b (g^2(x)-h^2(x))\,dx}{\int_a^b (g(x)-h(x))\,dx}\right) &\leq \frac{\int_a^b (g(x)-h(x))F(x, \frac{g(x)+h(x)}{2})\,dx}{\int_a^b (g(x)-h(x))\,dx} \\ &\leq \frac{\int_a^b (g(x)-h(x))F(x, \frac{g(x)+h(x)}{2})\,dx}{\int_a^b (g(x)-h(x))\,dx}. \end{aligned}$$

The proof is complete. □

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