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Convolution operators in the geometric function theory

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Abstract
The study of operators plays a vital role in mathematics. To define an operator using the convolution theory, and then study its properties, is one of the hot areas of current ongoing research in the geometric function theory and its related fields. In this survey-type article, we discuss historic development and exploit the strengths and properties of some differential and integral convolution operators introduced and studied in the geometric function theory. It is hoped that this article will be beneficial for the graduate students and researchers who intend to start work in this field.
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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$, centered at origin and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. Also, let $S \subset A$ be the class of functions which are univalent in E. The convolution or Hadamard product of two functions $f, g \in A$ is denoted by $f * g$ and is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \tag{1.2}$$

where $f(z)$ is given by (1.1), and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Note that the convolution of two functions is again analytic in E, *i.e.*, $(f * g)(z) \in A$.

For the complex parameters a, b and c with $c \neq 0, -1, -2, -3, \ldots$, the Gauss hypergeometric function ${}_2F_1(a, b, c; z)$ is defined as

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n \\ &= 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!} z^{n-1}, \end{aligned} \tag{1.3}$$

where 1 and 2 in the subscript of F merely signify the number of parameters in the numerator and the denominator of the coefficient of z^n respectively. Also, $(\alpha)_n$ is the Pochhammer symbol (or the shifted factorial) defined as

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if } n \neq 0. \end{cases} \tag{1.4}$$

The Pochhammer symbol is related to the factorial and the gamma functions by the relation

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}.$$

By S^*, C and K, we mean the subclasses of S consisting of starlike, with respect to the origin, convex and close-to-convex univalent functions in E respectively.

These classes of S^* and C are related to each other by the Alexander relation [1, 2]. Later Libera [3] introduced an integral operator and showed that these two classes are closed under this operator. Bernardi [4] gave a generalized operator and studied its properties. Ruscheweyh [5], Noor and Noor [6, 7], Noor [8] and many others, for example, [9–11], defined new operators and studied various classes of analytic and univalent functions generalizing a number of previously known classes and at times discovering new classes of analytic functions.

2 Convolution operators

The study of operators plays an important role in the geometric function theory. Many differential and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to understand the geometric properties of such operators better.

The importance of convolution in the theory of operators may be understood by the following set of examples given by Barnard and Kellogg [12].

Let $f \in A$ and $\Gamma_i : A \to A$ for each $i \in \{0,1,2,3\}$ be the linear operators defined as

$$\Gamma_0 f(z) = zf'(z) \quad \text{(Alexander differential [1])},$$

$$\Gamma_1 f(z) = \frac{(zf(z))'}{2} = \frac{f(z) + zf'(z)}{2} \quad \text{(Livingston [13])},$$

$$\Gamma_2 f(z) = \int_0^z \frac{f(\zeta)}{\zeta}\,d\zeta \quad \text{(Alexander integral [1, 2])},$$

$$\Gamma_3 f(z) = \frac{2}{z}\int_0^z f(\zeta)\,d\zeta \quad \text{(Libera [3])}.$$

Note that the first two of the above operators are differential and others are integral in nature. Now, each of these operators can be written as a convolution operator as follows:

$$\Gamma_i f(z) = (h_i * f)(z), \quad i \in \{0,1,2,3\},$$

where

$$h_0(z) = z + \sum_{n=2}^{\infty} nz^n = \frac{z}{(1-z)^2}, \qquad h_1(z) = z + \sum_{n=2}^{\infty} \frac{n+1}{2} z^n = \frac{z - \frac{z^2}{2}}{(1-z)^2},$$
$$h_2(z) = z + \sum_{n=2}^{\infty} \frac{1}{n} z^n = -\log(1-z), \qquad h_3(z) = z + \sum_{n=2}^{\infty} \frac{2}{n+1} z^n = \frac{-2[z + \log(1-z)]}{z}.$$

This shows how differential and integral operators may be written in terms of convolution of functions. Also, note that once we put these operators into convolution formalism, it becomes easy to conclude that the Alexander differential operator is the inverse of the Alexander integral operator, whereas the Livingston operator is the inverse of the Libera operator. For further discussion on the importance of the convolution operation, we recommend the reader to go through the classical work of Ruscheweyh [14].

Now, we give a brief survey of some convolution operators studied in the geometric function theory in chronological order and mention some related works.

2.1 The generalized Bernardi operator (1969)

Consider the operator $J_\eta : A \to A$ given by

$$J_\eta f(z) = \frac{1+\eta}{z^\eta} \int_0^z t^{\eta-1} f(t)\,dt, \quad \operatorname{Re}(\eta) > -1. \tag{2.1}$$

Note that the Alexander integral and Libera operators are special cases of J_η for $\eta = 0$ and $\eta = 1$ respectively. Now, $J_\eta f(z)$ can equivalently be put into a convolution formalism as

$$J_\eta f(z) = z\,{}_2F_1(1, 1+\eta, 2+\eta; z) * f(z), \tag{2.2}$$

where ${}_2F_1(1, 1+\eta, 2+\eta; z)$ is the Gaussian hypergeometric function given by (1.3). The operator J_η was introduced by Bernardi [4]. In [4], it was also shown that the classes S^* and C are closed under this operator, *i.e.*, the generalized Bernardi operator maps the classes of S^* and C onto the classes of S^* and C respectively. Some of other works on the Bernardi operator include [15] and [16] and references therein.

2.2 Ruscheweyh derivative operator (1975)

Using the technique of convolution, Ruscheweyh [5] defined the operator D^λ on the class of analytic functions A as

$$D^\lambda f(z) = \frac{z}{(1-z)^{\lambda+1}} * f(z), \quad \lambda \in \mathbb{R}, \lambda > -1. \tag{2.3}$$

For $\lambda = m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we obtain

$$D^m f(z) = \frac{z(z^{m-1} f(z))^{(m)}}{m!}. \tag{2.4}$$

The expression $D^m f(z)$ is called an mth-order Ruscheweyh derivative of $f(z)$. Note that $D^0 f(z) = f(z)$ which is identity operator, and $D^1 f(z) = zf'(z) = \Gamma_0$, the Alexander differential operator. It can also be shown that this operator is hypergeometric in nature as

$$D^{\lambda}f(z) = z\,{}_2F_1(\lambda+1,1,1;z) * f(z). \tag{2.5}$$

The following identity is easily established for the operator D^{λ}:

$$z\bigl(D^{\lambda}f\bigr)' = (\lambda+1)D^{\lambda+1}f - \lambda D^{\lambda}f.$$

Many authors, see, for example, [17–19], have used the Ruscheweyh operator to define and investigate the properties of certain known and new classes of analytic functions.

2.3 Carlson-Shaffer operator (1984)

Carlson and Shaffer [20] used the Hadamard product to define a linear operator $L(a,c) : A \to A$ by

$$L(a,c)f(z) = \varphi(a,c;z) * f(z), \tag{2.6}$$

where

$$\begin{aligned}\varphi(a,c;z) &= \sum_{n=0}^{\infty}\frac{(a)_n}{(c)_n}z^{n+1} = z + \sum_{n=2}^{\infty}\frac{(a)_{n-1}}{(c)_{n-1}}z^n, \quad c \neq 0,-1,-2,\ldots \\ &= z + \sum_{n=2}^{\infty}\frac{\Gamma(c)\Gamma(a+n-1)}{\Gamma(a)\Gamma(c+n-1)}z^n,\end{aligned} \tag{2.7}$$

is the incomplete beta function with $\varphi(a,c;z) \in A$. Using (1.3) and (2.7), we can establish a relation between hypergeometric and incomplete beta functions as

$$\varphi(a,c;z) = z\,{}_2F_1(a,1,c;z), \tag{2.8}$$

and hence

$$L(a,c)f(z) = z\,{}_2F_1(a,1,c;z) * f(z). \tag{2.9}$$

The Carlson-Shaffer operator maps A onto itself with $L(a,a)$ as the identity if $a \neq 0,-1,-2,\ldots$ and $L(c,a)$ for $a \neq 0,-1,-2,\ldots$ as the continuous inverse of $L(a,c)$, provided $c \neq 0,-1,-2,\ldots$. Moreover, it is known that

$$z\bigl[L(a,c)f(z)\bigr]' = aL(a+1,c)f(z) - (a-1)L(a,c)f(z), \quad c \neq 0,-1,-2,\ldots.$$

If we take $a = \lambda+1$, $c = 1$, then $L(\lambda+1,1)f(z) = D^{\lambda}f(z)$ which is the Ruscheweyh operator. Therefore, the Carlson-Shaffer operator generalizes the Ruscheweyh derivative operator defined in (2.3). Similarly, we observe that $L(2,1)f(z) = \Gamma_0$ and $L(3,2)f(z) = \Gamma_1$.

Recently, Shanmugam *et al.* [21] derived some sandwich theorems of certain subclasses of analytic functions associated with the Carlson-Shaffer operator.

2.4 Hohlov linear operator (1984)

By using the Gaussian hypergeometric function given by (1.3), Hohlov [22, 23] introduced a generalized convolution operator $H_{a,b,c}$ as

$$H_{a,b,c}f(z) = z\,{}_2F_1(a,b,c;z) * f(z), \tag{2.10}$$

and discussed some interesting geometrical properties exhibited by this operator. The three-parameter family of operators $H_{a,b,c}$ contains as special cases most of the known linear integral or differential operators. In particular, if $b = 1$ in (2.10), then $H_{a,1,c}$ reduces to the operator defined in (2.6), implying the Carlson-Shaffer operator a special case of the Hohlov operator. Similarly, it is straightforward to show that the Hohlov operator is also a generalization of Ruscheweyh and Bernardi operators.

It was shown in [23] that the linear hypergeometric operator $H_{1,b,c+\epsilon}$ maps the class S to itself for any positive ϵ if $b > 0$, $c > c_0(b) + b + 2$, where $c_0(b)$ is the greatest positive root of the equation

$$y^3 - 4by^2 - \left(5b^2 + b + 1\right)y - \left(2b^3 + b^2 - b\right) = 0.$$

Similarly, an interesting result describing the conditions when the Hohlov linear operator $H_{a,b,c}$ maps the class of convex functions C to that of univalent functions S was also reported in [23]. Recent findings of Mishra *et al.* [24], where they study class-mapping properties of the Hohlov operator, are worth mentioning.

2.5 Owa-Srivastava fractional differential operator (1987)

The fractional derivative of order α in the sense of Riemann-Liouville is defined as

$$D_z^\alpha f(z) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int_0^z \frac{f(\xi)}{(z-\xi)^\alpha}\,d\xi, \quad 0 \le \alpha < 1, \tag{2.11}$$

with $D_z^0 f(z) = f(z)$. We also assume $\log(z-\xi) \in \mathbb{R}$, *i.e.*, $z - \xi > 0$, to remove multiplicity of $(z-\xi)^{-\alpha}$ in the above integral. Fractional derivatives of higher order are defined by

$$D_z^{\beta+\alpha} f(z) = \frac{d^\beta}{dz^\beta} D_z^\alpha f(z), \quad \beta \in \mathbb{N}_0.$$

Using fractional derivative, Owa and Srivastava [25] introduced the operator $\Omega^\alpha : A \to A$, for $\alpha \ge 0$ by

$$\Omega^\alpha f(z) = \Gamma(2-\alpha) z^\alpha D_z^\alpha f(z), \quad \alpha \ne 2, 3, 4, \ldots,$$

which, upon using the infinite series form of $f(z)$ in (2.11), becomes

$$\begin{aligned}\Omega^\alpha f(z) &= z + \sum_{m=2}^{\infty} \frac{\Gamma(m+1)\Gamma(2-\alpha)}{\Gamma(m+1-\alpha)} a_m z^m \\ &= \varphi(2, 2-\alpha; z) * f(z).\end{aligned} \tag{2.12}$$

Using (2.8) in (2.12), we have

$$\Omega^{\alpha} f(z) = z\,{}_2F_1(2,1,2-\alpha;z) * f(z), \quad \alpha \neq 2,3,4,\ldots, \tag{2.13}$$

where $\varphi(2,2-\alpha;z)$ is the incomplete beta function given by (2.7), and Γ denotes the gamma function. Note that $\Omega^0 = f(z)$, the identity operator, and $\Omega^1 = zf'(z) = \Gamma_0$, the Alexander differential operator. See [9, 26] for a comprehensive discussion on this operator.

2.6 Noor integral operator (1999)

Analogous to Ruscheweyh derivative operator, Noor and Noor [6] and Noor [8] defined an operator as follows.

Let $f_n(z) = \frac{z}{(1-z)^{n+1}}$ and $f_n^{(-1)}(z)$ be defined as

$$f_n(z) * f_n^{(-1)}(z) = \frac{z}{(1-z)^2},$$

then

$$I_n f(z) = f_n^{(-1)}(z) * f(z). \tag{2.14}$$

It is straightforward to show that $I_0 f(z) = zf'(z)$ and $I_1 f(z) = f(z)$ are the Alexander differential operator and identity operator respectively. Thus the Noor operator is a generalization of the Alexander operator Γ_0 defined in the beginning of this section. Comparing the results obtained for I_0 and I_1 with those of D_0 and D_1 of the Ruscheweyh derivative operator, we note that $I_0 f(z) = D_1 f(z)$ and $I_1 f(z) = D_0 f(z)$. This reverse relation between the two operators provides us with a reason to call the Noor operator an integral operator. The operator I_n is called the nth-order Noor integral operator. The Noor integral operator in terms of convolution of a hypergeometric function may be given as

$$I_n f(z) = z\,{}_2F_1(2,1,n+1;z) * f(z). \tag{2.15}$$

Using (2.14), we have the following recursive relation for the Noor integral operator:

$$z(I_{n+1}f)' = (n+1)I_n f - nI_{n+1} f.$$

The study of the Noor integral operator and its related classes of analytic functions is still a topic of interest for many researchers, *cf.* [7, 10].

2.7 Dziok-Srivastava operator (1999)

The generalized hypergeometric function is merely an extension of the hypergeometric function (1.3) and is defined as

$${}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_q)_n}{(\beta_1)_n \cdots (\beta_s)_n n!} z^n,$$

where $q,s \in \mathbb{N}_0$ with $q \leq s+1$ and $\beta_i \neq 0,-1,-2,\ldots,$ for $i = 1,2,3,\ldots,s$. We again note that $z\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z) \in A$. Using the generalized hypergeometric function, Dziok and Srivastava [27–29] defined a convolution operator ${}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s):A\to A$ as

$$ {}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)=z\,{}_qF_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s;z)*f(z), \tag{2.16}$$

where $q\le s+1$ and $q,s\in\mathbb{N}_0$.

The following identities for the Dziok-Srivastava operator can easily be proved:

$$\begin{aligned} z\big[{}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)\big]' &= \alpha_1\,{}_qH_s(\alpha_1+1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)-(\alpha_1-1)\,{}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)\\ &= \alpha_2\,{}_qH_s(\alpha_1,\alpha_2+1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)-(\alpha_2-1)\,{}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z)\\ &\;\vdots\\ &= \alpha_q\,{}_qH_s(\alpha_1,\ldots,\alpha_q+1;\beta_1,\ldots,\beta_s)f(z)-(\alpha_q-1)\,{}_qH_s(\alpha_1,\ldots,\alpha_q;\beta_1,\ldots,\beta_s)f(z). \end{aligned}$$

For special values of parameters α's, β's, q and s, we obtain the operators of the generalized Bernardi, Ruscheweyh, Carlson-Shaffer, Hohlov, Owa-Srivastava, Noor and, as we shall see, of Choi-Saigo-Srivastava.

In [30] and [31], the authors have used this operator to produce subordination and superordination results.

2.8 Choi-Saigo-Srivastava operator (2002)

A natural generalization of the Noor integral operator is the operator $I_{\lambda,\mu}$ constructed in the following way:

$$I_{\lambda,\mu}f(z)=(f_{\lambda,\mu}*f)(z),\quad \lambda>-1,\mu>0; \tag{2.17}$$

where

$$\frac{z}{(1-z)^{\lambda+1}}*f_{\lambda,\mu}(z)=\frac{z}{(1-z)^{\mu}}.$$

The operator $I_{\lambda,\mu}$ was introduced and discussed by Choi, Saigo and Srivastava in [32]. In terms of a hypergeometric function, we can write it as

$$I_{\lambda,\mu}f(z)=z\,{}_2F_1(\mu,1,\lambda+1;z)*f(z). \tag{2.18}$$

It is obvious that for $\mu=2$ this operator reduces to the Noor integral operator defined in (2.14), whereas for $\mu=2$ and $\lambda=1-\alpha$, we get the Owa-Srivastava operator given by (2.12). Also note that $I_{0,2}f(z)=zf'(z)$ and $I_{1,2}f(z)=f(z)$, and that is why this operator is termed as integral in literature. Using (2.17), the following identities can easily be verified:

$$\begin{aligned} z\big[I_{\lambda+1,\mu}f(z)\big]' &= (\lambda+1)I_{\lambda,\mu}f(z)-\lambda I_{\lambda+1,\mu}f(z),\\ z\big[I_{\lambda,\mu}f(z)\big]' &= \mu I_{\lambda,\mu+1}f(z)-(\mu-1)I_{\lambda,\mu}f(z). \end{aligned}$$

In [33] and [34], the authors have discussed some interesting properties of analytic functions associated with the Choi-Saigo-Srivastava operator.

2.9 Srivastava-Attiya operator (2007)

The generalized Hurwitz-Lerch zeta function $\phi(\mu, b; z)$ is defined by

$$\begin{aligned}\phi(\mu, b; z) &= \sum_{n=0}^{\infty} \frac{z^n}{(n+b)^\mu} \\ &= b^{-\mu} + \frac{z}{(1+b)^\mu} + \sum_{n=2}^{\infty} \frac{z^n}{(n+b)^\mu},\end{aligned} \tag{2.19}$$

where $b \in \mathbb{C}$ with $b \neq 0, -1, -2, -3, \ldots$, $\mu \in \mathbb{C}$, $\operatorname{Re}(\mu) > 1$ and $|z| < 1$.

Srivastava and Attiya [35] introduced a family of linear operators $J_{\mu,b} : A \to A$ by the Hadamard product of the Hurwitz-Lerch zeta function with an analytic function as

$$J_{\mu,b} f(z) = G_{\mu,b} * f(z), \tag{2.20}$$

where $b \in \mathbb{C}$ with $b \neq 0, -1, -2, -3, \ldots$, $\mu \in \mathbb{C}$, $z \in E$ and $G_{\mu,b} \in A$ given by

$$\begin{aligned}G_{\mu,b} &= (1+b)^\mu \left[\phi(\mu, b; z) - b^{-\mu}\right] \\ &= z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^\mu z^n.\end{aligned} \tag{2.21}$$

Note that $J_{0,b}$ and $J_{-\mu,b}$ give the identity and inverse operators of $J_{\mu,b}$ respectively. The following recursive relation can easily be obtained using (2.20) and (2.21):

$$z\left[J_{\mu,b} f(z)\right]' = (1+b) J_{\mu-1,b} f(z) - b J_{\mu,b} f(z).$$

This operator contains many known operators as its special cases for different values of μ and b; see [35] for a complete list of such operators. This paper by Srivastava and Attiya discusses some interesting subordination results for this operator as well. Whereas [36] discusses its applications on strongly starlike and convex functions.

2.10 The multiplier fractional differential operator (2008)

Al-Oboudi and Al-Amoudi [37] extended the Owa-Srivastava fractional differential operator [25] and proposed a multiplier fractional differential operator.

The multiplier fractional differential operator $D_\lambda^{n,\alpha}$ for $\lambda \geq 0$, $n \in \mathbb{N}_0$ and $\alpha \geq 0$ is defined as follows:

$$\begin{aligned}D_\lambda^{0,\alpha} f(z) &= f(z); \\ D_\lambda^{1,\alpha} f(z) &= (1-\lambda)\Omega^\alpha f(z) + \lambda z\left(\Omega^\alpha f(z)\right)' := D_\lambda^\alpha f(z), \quad \alpha \neq 2, 3, 4, \ldots; \\ D_\lambda^{2,\alpha} f(z) &= D_\lambda^\alpha\left(D_\lambda^{1,\alpha} f(z)\right); \\ &\vdots \\ D_\lambda^{n,\alpha} f(z) &= D_\lambda^\alpha\left(D_\lambda^{n-1,\alpha} f(z)\right), \quad n \in \mathbb{N}.\end{aligned}$$

Using the power series expansion of $f(z)$, we can write

$$D_\lambda^{n,\alpha} f(z) = z + \sum_{m=2}^{\infty} \Psi_{m,n}(\alpha,\lambda) a_m z^m, \quad n \in \mathbb{N}_0, \alpha \neq 2,3,4,\ldots, \tag{2.22}$$

where

$$\Psi_{m,n}(\alpha,\lambda) = \left[\frac{\Gamma(m+1)\Gamma(2-\alpha)}{\Gamma(m+1-\alpha)}\big(1+\lambda(m-1)\big)\right]^n. \tag{2.23}$$

Also, in terms of the convolution of an incomplete beta function, we may put (2.22) in the following form:

$$D_\lambda^{n,\alpha} f(z) = \big[\underbrace{\varphi(2,2-\alpha;z) * g_\lambda(z) * \cdots * \varphi(2,2-\alpha;z) * g_\lambda(z)}_{n\text{-times}}\big] * f(z), \tag{2.24}$$

or equivalently, using a hypergeometric function

$$D_\lambda^{n,\alpha} f(z) = \big[\underbrace{z\,{}_2F_1(2,1,2-\alpha;z) * g_\lambda(z) * \cdots * z\,{}_2F_1(2,1,2-\alpha;z) * g_\lambda(z)}_{n\text{-times}}\big] * f(z), \tag{2.25}$$

where $g_\lambda(z) = \frac{z-(1-\lambda)z^2}{(1-z)^2}$. Note that for $n = 1$ and $\lambda = 0$, $g_\lambda(z)$ reduces to the identity of a convolution operation, and $D_\lambda^{n,\alpha}$ becomes the Owa-Srivastava operator (2.12).

In [38], some subordination results have been derived for this fractional operator.

3 Concluding remarks

The operation of convolution and convolution operators are the topics of great interest for researchers. Studying operators expressed in terms of convolution helps us explore their geometrical properties. We observe that most of the convolution operators discussed in this paper are hypergeometric operators. We also note that most of the operators discussed in this article are merely special cases of the Hohlov operator which is a special case of the Dziok-Srivastava operator.

Many authors have used these operators on previously known classes of analytic and univalent functions to produce new classes and to investigate several interesting properties of new classes. Also, taking inspiration from these operators, new operators were defined on the classes of multivalent [11, 39], meromorphic [40, 41] and harmonic functions [42, 43].

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
The first author, who currently is a PhD student under supervision of the second author (co-supervisor), and third author (main supervisor) jointly worked on deriving the results. All authors read and approved the final manuscript.

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