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On an inequality suggested by Littlewood

Peng Gao

Correspondence: penggao@ntu.edu.sg
Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 637371 Singapore

Abstract
We study an inequality suggested by Littlewood, our result refines a result of Bennett. 2000 *Mathematics Subject Classification*. Primary 26D15.

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Introduction

In connection with work on the general theory of orthogonal series, Littlewood [1] raised some problems concerning elementary inequalities for infinite series. One of them asks to decide whether an absolute constant K exists such that for any non-negative sequence (a_n) with $A_n = \sum_{k=1}^{n} a_k$,

$$\sum_{n=1}^{\infty} a_n A_n^2 \left(\sum_{k=n}^{\infty} a_k^{3/2} \right)^2 \leq K \sum_{n=1}^{\infty} a_n^2 A_n^4. \tag{1.1}$$

The above problem was solved by Bennett [2], who proved the following more general result:

Theorem 1.1 ([2, Theorem 4]). Let $p \geq 1$, $q > 0$, $r > 0$ *satisfying* $(p(q + r) - q)/p \geq 1$ *be fixed. Let* $K(p, q, r)$ *be the best possible constant such that for any non-negative sequence* (a_n) *with* $A_n = \sum_{k=1}^{n} a_k$,

$$\sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{k=n}^{\infty} a_k^{1+p/q} \right)^r \leq K(p, q, r) \sum_{n=1}^{\infty} \left(a_n^p A_n^q \right)^{1+r/q}. \tag{1.2}$$

Then

$$K(p, q, r) \leq \left(\frac{p(q+r) - q}{p} \right)^r.$$

The special case $p = 1$, $q = r = 2$ in (1.2) leads to inequality (1.1) with $K = 4$ and Theorem 1.1 implies that $K(p, q, r)$ is finite for any $p \geq 1$, $q > 0$, $r > 0$ satisfying $(p(q + r) - q)/p \geq 1$, a fact we shall use implicitly throughout this article. We note that Bennett only proved Theorem 1.1 for $p, q, r \geq 1$ but as was pointed out in [3], Bennett's proof actually works for the p, q, r's satisfying the condition in Theorem 1.1. Another proof of inequality (1.2) for the special case $r = q$ was provided by Bennett [4] and a close look at the proof there shows that it in fact can be used to establish Theorem 1.1.

On setting $p = 2$ and $q = r = 1$ in (1.2), and interchanging the order of summation on the left-hand side of (1.2), we deduce the following

$$\sum_{n=1}^{\infty} a_n^3 \sum_{k=1}^{n} a_k^2 A_k \le \frac{3}{2} \sum_{n=1}^{\infty} a_n^4 A_n^2. \tag{1.3}$$

The constant in (1.3) was improved to be $2^{1/3}$ in [5] and the following more general result was given in [6]:

Theorem 1.2 ([6, Theorem 2]). *Let* $p, q \ge 1$, $r > 0$ *be fixed satisfying* $r(p - 1) \le 2(q - 1)$. *Set*

$$\alpha = \frac{(p-1)(q+r) + p^2 + 1}{p+1}, \quad \beta = \frac{2q + 2r + p - 1}{p+1}, \quad \delta = \frac{q+r-1}{p+q+r}.$$

Then for any non-negative sequence (a_n) *with* $A_n = \sum_{k=1}^{n} a_k$,

$$\sum_{n=1}^{\infty} a_n^p \sum_{k=1}^{n} a_k^q A_k^r \le 2^{\delta} \sum_{n=1}^{\infty} a_n^{\alpha} A_n^{\beta}. \tag{1.4}$$

Note that inequality (1.3) with constant $2^{1/3}$ corresponds to the case $p = 3$, $q = 2$, $r = 1$ in (1.4). In [7], an even better constant was obtained but the proof there is incorrect. In [3,6,7], results were also obtained concerning inequality (1.2) under the extra assumption that the sequence (a_n) is non-decreasing.

The exact value of $K(p, q, r)$ is not known in general. But note that $K(1, q, 1) = 1$ as it follows immediately from Theorem 1.1 that $K(1, q, 1) \le 1$ while on the other hand on setting $a_1 = 1$, $a_n = 0$, $n \ge 2$ in (1.2) that $K(1, q, 1) \ge 1$. Therefore, we may restrict our attention on (1.2) for p, r's not both being 1. In this article, it is our goal to improve the result in Theorem 1.1 in the following

Theorem 1.3. *Let* $p \ge 1$, $q > 0$, $r \ge 1$ *be fixed with* p, r *not both being* 1. *Under the same notions of Theorem 1.1, inequality* (1.2) *holds when* $q + r - q/p \ge 2$ *with*

$$K(p, q, r) \le K\left(p\left(1 + \frac{r-1}{q}\right), q + r - 1, 1\right)\left(\frac{p(q+r) - q}{p}\right)^{r-1}.$$

When $1 \le q + r - q/p \le 2$, *inequality* (1.2) *holds with*

$$K(p, q, r) \le \left(K\left(p\left(1 + \frac{r-1}{q}\right), q + r - 1, 1\right)\right)^{r - \frac{p(r-1)}{pq + p(r-1) - q}} \left(\frac{p(q+r) - q}{p}\right)^{\frac{p(r-1)}{pq + p(r-1) - q}}.$$

Moreover, for any $p \ge 1$, $q > 0$,

$$K(p, q, 1) \le \min_{\delta}\left(\frac{p(q+1) - q}{p}, C(p, q, \delta)\right), \tag{1.5}$$

where

$$C(p, q, \delta) = \left(\delta\left(1 + \frac{p}{q(p-1)}\right)\left(1 + \frac{1}{1/(p-1) + \delta(1 + p/(q(p-1))) - 1}\right)\right)^{\delta}, \tag{1.6}$$

and the minimum in (1.5) *is taken over the* δ*'s satisfying*

$$\frac{q(p-1)}{p(q+1) - q} \le \delta \le 1. \tag{1.7}$$

On considering the values of $C(p, q, \delta)$ for $\delta = 1$ and $\delta = q(p - 1)/(p(q + 1) - q)$, we readily deduce from Theorem 1.3 the following

Corollary 1.1. *Let* $p \geq 1$, $q > 0$ *be fixed. Let* $K(p, q, r)$ *be the best possible constant such that inequality* (1.2) *holds for any non-negative sequence* (a_n). *Then*

$$K(p,\ q,\ 1) \leq \min\left(\frac{p(q+1)-q}{p}, p^{\frac{(p-1)q}{(p-1)q+p}}, \left(1+\frac{(p-1)q}{p+q}\right)\left(1+\frac{p}{q(p-1)}\right)\right). \quad (1.8)$$

We note that Theorem 1.3 together with Lemma 2.4 below shows that a bound for $K(p, q, r)$ with $p \geq 1$, $q > 0$, $r > 0$ satisfying $(p(q + r) q)/p \geq 1$ can be obtained by a bound of $K(p(1 + (r - 1)/q), q + r - 1, 1)$ and as (1.8) implies that $K(p(1 + (r - 1)/q), q + r - 1, 1) \leq (p(q + r) - q)/p$, it is easy to see that the assertion of Theorem 1.1 follows from the assertions of Theorem 1.3 and Lemma 2.4.

We point out here that among the three expressions on the right-hand side of (1.8), each one is likely to be the minimum. For example, the middle one becomes the minimum when $p = 2$, $q = 1$ while it is easy to see that the last one becomes the minimum for $p = q$ large enough and the first one becomes the minimum when q is being fixed and $p \to \infty$. Moreover, it can happen that the minimum value in (1.5) occurs at a δ other than $q(p - 1)/(p(q + 1) - q)$, 1. For example, when $p = q = 6$, the bound (1.8) gives $K(6, 6, 1) \leq 21/5$ while one checks easily that $C(6, 6, 1.15/1.2) < 21/5$. We shall not worry about determining the precise minimum of (1.5) in this article.

We note that the special case $p = 1$, $q = r = 2$ of Theorem 1.3 leads to the following improvement on Bennet's result on the constant K of inequality (1.1):

Corollary 1.2. *Inequality* (1.1) *holds with* $K = \sqrt{6}$.

A few Lemmas

Lemma 2.1. *Let* $d \geq c > 1$ *and* (λ_n) *be a non-negative sequence with* $\lambda_1 > 0$. *Let* $\Lambda_n = \sum_{k=1}^{n} \lambda_k$. *Then for all non-negative sequences* (x_n),

$$\sum_{n=1}^{\infty} \lambda_n \Lambda_n^{-c} \left(\sum_{k=1}^{n} \lambda_k x_k\right)^d \leq \left(\frac{d}{c-1}\right)^d \sum_{n=1}^{\infty} \lambda_n \Lambda_n^{d-c} x_n^d.$$

The constant is best possible.

The above lemma is the well-known Copson's inequality [8, Theorem 1.1], see also Corollary 3 to Theorem 2 of [2].

Lemma 2.2. *Let* $p < 0$. *For any non-negative sequence* (a_n) *with* $a_1 > 0$ *and* $A_n = \sum_{k=1}^{n} a_k$, *we have for any* $n \geq 1$,

$$\sum_{k=n}^{\infty} a_k A_k^{p-1} \leq \left(1 - \frac{1}{p}\right) A_n^p. \quad (2.1)$$

Proof. We start with the inequality $x^p - px + p - 1 \geq 0$. By setting $x = A_{k-1}/A_k$ for $k \geq 2$, we obtain

$$A_{k-1}^p - pA_{k-1}A_k^{p-1} + (p-1)A_k^p \geq 0.$$

Replacing A_{k-1} in the middle term of the left-hand side expression above by A_k-a_k and simplifying, we obtain

$$A_{k-1}^p - A_k^p \geq -pa_k A_k^{p-1}.$$

Upon summing, we obtain

$$\sum_{k=n+1}^{\infty} a_k A_k^{p-1} \le -\frac{1}{p} A_n^p.$$

Inequality (2.1) follows from above upon noting that $a_n A_n^{p-1} \le A_n^p$. □

Lemma 2.3. *Let* $p \ge 1$, $q > 0$, $r \ge 1$ *be fixed with* p, r *not both being* 1. *Under the same notions of Theorem 1.1, we have*

$$K(p,q,r) \le \left(K\left(p\left(1+\frac{r-1}{q}\right), q+r-1, 1\right)\right)^{\frac{p-1}{p+p(r-1)/q-1}} \left(K\left(1, \frac{q}{p}, \frac{p(q+r)-q}{p}\right)\right)^{\frac{p(r-1)}{pq+p(r-1)-q}}.$$

Proof. As it is easy to check the assertion of the lemma holds when $p = 1$ or $r = 1$, we may assume $p > 1$, $r > 1$ here. We set

$$\alpha = \frac{p-1}{p+p(r-1)/q-1}, \quad \beta = \alpha\left(1+\frac{r-1}{q}\right), \quad b_n = a_n A_n^{q/p}, \quad c_n = \sum_{k=n}^{\infty} a_k^{1+p/q}.$$

Note that we have $0 < \alpha < 1$ as $p > 1$, $r > 1$ here. By Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{k=n}^{\infty} a_k^{1+p/q}\right)^r &= \sum_{n=1}^{\infty} b_n^p c_n^r = \sum_{n=1}^{\infty} b_n^{p\beta} c_n^{\alpha} \cdot b_n^{p(1-\beta)} c_n^{r-\alpha} \\ &\le \left(\sum_{n=1}^{\infty} b_n^{p\beta/\alpha} c_n\right)^{\alpha} \cdot \left(\sum_{n=1}^{\infty} b_n^{p(1-\beta)/(1-\alpha)} c_n^{(r-\alpha)/(1-\alpha)}\right)^{1-\alpha} \\ &= \left(\sum_{n=1}^{\infty} b_n^{p(1+(r-1)/q)} c_n\right)^{\frac{p-1}{p+p(r-1)/q-1}} \cdot \left(\sum_{n=1}^{\infty} b_n c_n^{(p(q+r)-q)/p}\right)^{\frac{p(r-1)}{pq+p(r-1)-q}}.\end{aligned} \tag{2.2}$$

The assertion of the lemma now follows on applying inequality (1.2) to both factors of the last expression above.□

Lemma 2.4. *Let* $p \ge 1$, $q > 0$, $0 < r \le 1$ *be fixed satisfying* $(p(q + r) - q)/p \ge 1$. *Under the same notions of Theorem 1.1, we have*

$$K(p,\ q,\ r)| \le \left(K\left(p\left(1+\frac{r-1}{q}\right), q+r-1, 1\right)\right)^{r}. \tag{2.3}$$

Proof. We may assume $0 < r < 1$ here. We set

$$\alpha = 1-r, \quad \beta = \alpha\left(1+\frac{r}{q}\right), \quad b_n = a_n A_n^{q/p}, \quad c_n = \sum_{k=n}^{\infty} a_k^{1+p/q}.$$

Note that we have $0 < \alpha < 1$. By Hölder's inequality, we have

$$\begin{aligned}\sum_{n=1}^{\infty} a_n^p A_n^q \left(\sum_{k=n}^{\infty} a_k^{1+p/q}\right)^r &= \sum_{n=1}^{\infty} b_n^p c_n^r = \sum_{n=1}^{\infty} b_n^{p\beta} \cdot b_n^{p(1-\beta)} c_n^r \\ &\le \left(\sum_{n=1}^{\infty} b_n^{p\beta/\alpha}\right)^{\alpha} \left(\sum_{n=1}^{\infty} b_n^{p(1-\beta)/(1-\alpha)} c_n^{r/(1-\alpha)}\right)^{1-\alpha} \\ &= \left(\sum_{n=1}^{\infty} b_n^{p(1+r/q)}\right)^{1-r} \left(\sum_{n=1}^{\infty} b_n^{p(1+(r-1)/q)} c_n\right)^{r}.\end{aligned}$$

The assertion of the lemma now follows on applying inequality (1.2) to the second factor of the last expression above.□

Proof of Theorem 1.3

We obtain the proof of Theorem 1.3 via the following two lemmas:

Lemma 3.1. *Let $p \geq 1$, $q > 0$ be fixed. Under the same notions of Theorem 1.1, inequality* (1.2) *holds when $r = 1$ with $K(p, q, 1)$ bounded by the right-hand side expression of* (1.5).

Proof. We may assume that only finitely many a_n's are positive, say $a_n = 0$ whenever $n > N$. We may also assume $a_1 > 0$. As the case $p = 1$ of the lemma is already contained in Theorem 1.1, we may further assume $p > 1$ throughout the proof. Moreover, even though the assertion that $K(p, q, 1) \leq (p(q + 1) - q)/p$ is already given in Theorem 1.1, we include a new proof here.

We recast the left-hand side expression of (1.2) as

$$\begin{aligned}\sum_{n=1}^{N} a_n^p A_n^q \sum_{k=n}^{N} a_k^{1+p/q} &= \sum_{n=1}^{N} a_n^{1+p/q} \sum_{k=1}^{n} a_k^p A_k^q \\ &= \sum_{n=1}^{N} (a_n^p A_n^q)^{\theta(1+1/q)} \cdot a_n^{1+p/q} (a_n^p A_n^q)^{-\theta(1+1/q)} \sum_{k=1}^{n} a_k^p A_k^q \\ &\leq \left(\sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q}\right)^{\theta} \left(\sum_{n=1}^{N} a_n^{(1+p/q)/(1-\theta)} (a_n^p A_n^q)^{-\theta(1+1/q)/(1-\theta)} \left(\sum_{k=1}^{n} a_k^p A_k^q\right)^{1/(1-\theta)}\right)^{1-\theta} \\ &= \left(\sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q}\right)^{\theta} \left(\sum_{n=1}^{N} a_n A_n^{-p(q+1)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^p A_k^q\right)^{(p(q+1)-q)/(q(p-1))}\right)^{1-\theta},\end{aligned} \tag{3.1}$$

where we set

$$\theta = \frac{p}{p(q+1) - q},$$

so that $0 < \theta < 1$ and the inequality in (3.1) follows from an application of Hölder's inequality.

Thus, to prove Theorem 1.3, it suffices to show that

$$\sum_{n=1}^{N} a_n A_n^{-p(q+1)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^p A_k^q\right)^{(p(q+1)-q)/(q(p-1))} \leq K_1(p,\ q) \sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q}, \tag{3.2}$$

where

$$K_1(p,\ q) = \min_{\delta}\left(\left(\frac{p(q+1) - q}{p}\right)^{1/(1-\theta)},\ C(p,\ q,\ \delta)^{1/(1-\theta)}\right),$$

where $C(p, q, \delta)$ is defined as in (1.6) and the minimum is taken over the δ 's satisfying (1.7).

Note first we have

$$\begin{aligned}&\sum_{n=1}^{N} a_n A_n^{-p(q+1)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^p A_k^q\right)^{(p(q+1)-q)/(q(p-1))} \\ &\leq \sum_{n=1}^{N} a_n A_n^{-p(q+1)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^p A_k^{q-q/(p(q+1)-q)} A_n^{q/(p(q+1)-q)}\right)^{(p(q+1)-q)/(q(p-1))} \\ &= \sum_{n=1}^{N} a_n \left(\sum_{k=1}^{n} \frac{a_k}{A_n} \left(a_k^{p-1} A_k^{q-q/(p(q+1)-q)}\right)\right)^{(p(q+1)-q)/(q(p-1))} \\ &\leq \left(\frac{p(q+1) - q}{p}\right)^{(p(q+1)-q)/(q(p-1))} \sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q},\end{aligned}$$

where the last inequality above follows from Lemma 2.1 by setting $d = c = (p(q + 1)\ q)/(q(p - 1))$, $\lambda_n = a_n$, $x_n = a_n^{p-1} A_n^{q-q/(p(q+1)-q)}$ there. This establishes (3.2) with

$$K_1(p,\ q) = \left(\frac{p(q+1)-q}{p}\right)^{1/(1-\theta)}.$$

Next, we use the idea in [5] (see also [6]) to see that for any $0 < \delta \le 1$,

$$\begin{aligned}
&\sum_{n=1}^{N} a_n A_n^{-p(q+1)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^p A_k^q\right)^{(p(q+1)-q)/(q(p-1))} \\
&= \sum_{n=1}^{N} a_n A_n^{-1/(p-1)} \left(\sum_{k=1}^{n} \frac{a_k}{A_n} a_k^{p-1} A_k^q\right)^{(p(q+1)-q)/(q(p-1))} \\
&= \sum_{n=1}^{N} a_n A_n^{-1/(p-1)} \left(\sum_{k=1}^{n} \frac{a_k}{A_n} \left(a_k^{(p-1)/\delta} A_k^{q/\delta}\right)^{\delta}\right)^{(p(q+1)-q)/(q(p-1))} \\
&\le \sum_{n=1}^{N} a_n A_n^{-1/(p-1)} \left(\sum_{k=1}^{n} \frac{a_k}{A_n} a_k^{(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))}.
\end{aligned} \tag{3.3}$$

We now further require that

$$\frac{q(p-1)}{p(q+1)-q} < \delta \le 1,$$

then on setting for $1 \le n \le N$,

$$S_n = \sum_{k=n}^{N} a_k A_k^{-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))}, \quad T_n = \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))},$$

we have by partial summation, with $T_0 = 0$,

$$\begin{aligned}
&\sum_{n=1}^{N} a_n A_n^{-1/(p-1)} \left(\sum_{k=1}^{n} \frac{a_k}{A_n} a_k^{(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))} \\
&= \sum_{n=1}^{N} a_n A_n^{-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))} \\
&= \sum_{n=1}^{N} S_n (T_n - T_{n-1}) \\
&\le \delta\left(1 + \frac{p}{q(p-1)}\right) \sum_{n=1}^{N} S_n \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))-1} a_n^{1+(p-1)/\delta} A_n^{q/\delta} \\
&\le \delta\left(1 + \frac{p}{q(p-1)}\right)\left(1 + \frac{1}{1/(p-1) + \delta(1 + p/(q(p-1))) - 1}\right) \\
&\quad \cdot \sum_{n=1}^{N} \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))-1} a_n^{1+(p-1)/\delta} A_n^{q/\delta+1-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))},
\end{aligned} \tag{3.4}$$

where for the first inequality in (3.4), we have used the bound

$$T_n - T_{n-1} \le \delta\left(1 + \frac{p}{q(p-1)}\right)\left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))-1} a_n^{1+(p-1)/\delta} A_n^{q/\delta},$$

by the mean value theorem and for the second inequality in (3.4), we have used the bound

$$S_n \le \left(1 + \frac{1}{1/(p-1) + \delta(1 + p/(q(p-1))) - 1}\right) A_n^{1-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))},$$

by Lemma 2.2.

Now, we set

$$P = \frac{\delta(p(q+1)-q)}{\delta(p(q+1)-q) - q(p-1)}, \quad Q = \frac{\delta(p(q+1)-q)}{q(p-1)},$$

so that P, $Q > 1$, and $1/P + 1/Q = 1$. We then have, by Hölder's inequality,

$$\begin{aligned}
&\sum_{n=1}^{N} \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))-1} a_n^{1+(p-1)/\delta} A_n^{q/\delta+1-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))} \\
&= \sum_{n=1}^{N} \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{Q/P} a_n^{1/P} A_n^{-(1/(p-1)+Q)/P} \cdot a_n^{1/Q+(p-1)/\delta} A_n^{q/\delta+1-(1/(p-1)+Q)/Q} \\
&\le \left(\sum_{n=1}^{N} a_n A_n^{-1/(p-1)-\delta(p(q+1)-q)/(q(p-1))} \left(\sum_{k=1}^{n} a_k^{1+(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))}\right)^{1/P} \\
&\quad \cdot \left(\sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q}\right)^{1/Q}.
\end{aligned} \tag{3.5}$$

It follows from inequalities (3.4) and (3.5) that

$$\begin{aligned}
&\sum_{n=1}^{N} a_n A_n^{-1/(p-1)} \left(\sum_{k=1}^{n} \frac{a_k}{A_n} a_k^{(p-1)/\delta} A_k^{q/\delta}\right)^{\delta(p(q+1)-q)/(q(p-1))} \\
&\le \left(\delta\left(1 + \frac{p}{q(p-1)}\right)\left(1 + \frac{1}{1/(p-1) + \delta(1 + p/(q(p-1))) - 1}\right)\right)^{Q} \\
&\quad \cdot \sum_{n=1}^{N} (a_n^p A_n^q)^{1+1/q}.
\end{aligned}$$

One sees easily that the above inequality also holds when $\delta = q(p - 1)/(p(q + 1) - q)$. Combining the above inequality with (3.3), we see this establishes (3.2) with

$$K_1(p, q) = \min_{\delta} \left(C(p, q, \delta)^{1/(1-\theta)}\right),$$

where $C(p, q, \delta)$ is defined as in (1.6) and the minimum is taken over the δ 's satisfying (1.7) and this completes the proof of Lemma 3.1. □

Lemma 3.2. *Let $p = 1$, $q > 0$, $r \ge 1$ be fixed. Under the same notions of Theorem 1.1, we have*

$$K(1, q, r) \le \begin{cases} r^{r-1} K(1 + (r-1)/q, q + r - 1, 1), & r \ge 2; \\ r(K(1 + (r-1)/q, q + r - 1, 1))^{r-1}, & 1 \le r \le 2. \end{cases}$$

Proof. We may assume $a_n = 0$ whenever $n > N$. In this case, on setting

$$b_n = a_n A_n^q, \ c_n = \sum_{k=n}^{N} a_k^{1+1/q}, \ B_n = \sum_{k=1}^{n} b_k,$$

the left-hand side expression of (1.2) becomes

$$\sum_{n=1}^{N} b_n c_n^r.$$

Note that as $r \geq 1$, we have the following bounds:

$$B_n \leq A_n^{1+q},\ c_n^r - c_{n+1}^r \leq r c_n^{r-1} a_n^{1+1/q}.$$

We then apply partial summation together with the bounds above to obtain (with $B_0 = c_{N+1} = 0$)

$$\sum_{n=1}^{N} b_n c_n^r = \sum_{n=1}^{N} (B_n - B_{n-1}) c_n^r = \sum_{n=1}^{N} B_n (c_n^r - c_{n+1}^r) \leq r \sum_{n=1}^{N} a_n^{1+1/q} A_n^{1+q} c_n^{r-1}. \tag{3.6}$$

When $r \geq 2$, we apply inequality (2.2) to see that

$$\sum_{n=1}^{N} a_n^{1+1/q} A_n^{1+q} c_n^{r-1} \leq \left(\sum_{n=1}^{N} a_n^{1+(r-1)/q} A_n^{q+r-1} \sum_{k=n}^{N} a_k^{1+1/q} \right)^{\frac{1}{r-1}} \cdot \left(\sum_{n=1}^{N} b_n c_n^r \right)^{\frac{r-2}{r-1}}.$$

Combining this with inequality (3.6), we see that this implies that

$$\sum_{n=1}^{N} b_n c_n^r \leq r^{r-1} \sum_{n=1}^{N} a_n^{1+(r-1)/q} A_n^{q+r-1} \sum_{k=n}^{N} a_k^{1+1/q}.$$

The assertion of the lemma for $r \geq 2$ now follows on applying inequality (1.2) to the right-hand side expression above.

When $1 \leq r \leq 2$, we apply inequality (2.3) in (3.6) to see that

$$K(1,\ q,\ r) \leq rK(1 + 1/q,\ 1 + q,\ r - 1) \leq r(K(1 + (r - 1)/q,\ q + r - 1, 1))^{r-1}.$$

The assertion of the lemma for $1 \leq r \leq 2$ now follows and this completes the proof. □

Now, to establish Theorem 1.3, it suffices to apply Lemma 2.3 with the observation that when $q + r - q/p \geq 2$, Lemma 3.2 implies that

$$K\left(1, \frac{q}{p}, \frac{p(q+r)-q}{p}\right) \leq \left(\frac{p(q+r)-q}{p}\right)^{\frac{p(q+r-1)-q}{p}} K\left(p\left(1+\frac{r-1}{q}\right),\ q+r-1, 1\right),$$

while when $1 \leq q + r - q/p \leq 2$, Lemma 3.2 implies that

$$K\left(1, \frac{q}{p}, \frac{p(q+r)-q}{p}\right) \leq \left(\frac{p(q+r)-q}{p}\right) \left(K\left(p\left(1+\frac{r-1}{q}\right),\ q+r-1, 1\right)\right)^{\frac{p(q+r-1)-q}{p}}.$$

The bound for $K(p, q, 1)$ follows from Lemma 3.1 and this completes the proof of Theorem 1.3.

Further discussions

We now look at inequality (1.2) in a different way. For this, we define for any non-negative sequence (a_n) and any integers $N \geq n \geq 1$,

$$A_{n,N} = \sum_{k=n}^{N} a_k, \ A_{n,\infty} = \sum_{k=n}^{\infty} a_k.$$

We then note that in order to establish inequality (1.2), it suffices to show that for any integer $N \geq 1$, we have

$$\sum_{n=1}^{N} a_n^p A_n^q \left(\sum_{k=n}^{N} a_k^{1+p/q} \right)^r \leq K(p,\ q,\ r) \sum_{n=1}^{N} \left(a_n^p A_n^q \right)^{1+r/q}.$$

Upon a change of variables: $a_n \ \alpha \ a_{N\ -\ n+1}$ and recasting, we see that the above inequality is equivalent to

$$\sum_{n=1}^{N} a_n^p A_{n,N}^q \left(\sum_{k=1}^{n} a_k^{1+p/q} \right)^r \leq K(p,\ q,\ r) \sum_{n=1}^{N} \left(a_n^p A_{n,N}^q \right)^{1+r/q}.$$

On letting $N \to \mapsto \infty$, we see that inequality (1.2) is equivalent to the following inequality:

$$\sum_{n=1}^{\infty} a_n^p A_{n,\infty}^q \left(\sum_{k=1}^{n} a_k^{1+p/q} \right)^r \leq K(p,\ q,\ r) \sum_{n=1}^{\infty} \left(a_n^p A_{n,\infty}^q \right)^{1+r/q}. \tag{4.1}$$

Here $K(p, q, r)$ is also the best possible constant such that inequality (4.1) holds for any non-negative sequence (a_n).

We point out that one can give another proof of Theorem 1.3 by studying (4.1) directly. As the general case $r \geq 1$ can be reduced to the case $r = 1$ in a similar way as was done in the proof of Theorem 1.3 in Sect. 3, one only needs to establish the upper bound for $K(p, q, 1)$ given in (1.5). For this, one can use an approach similar to that taken in Sect. 3, in replacing Lemmas 2.1 and 2.2 by the following lemmas. Due to the similarity, we shall leave the details to the reader.

Lemma 4.1. *Let* $d \geq c > 1$ *and* (λ_n) *be a positive sequence with* $\sum_{k=1}^{\infty} \lambda_k < \infty$. *Let* $\Lambda_n^* = \sum_{k=n}^{\infty} \lambda_k$. *Then for all non-negative sequences* (x_n),

$$\sum_{n=1}^{\infty} \lambda_n (\Lambda_n^*)^{-c} \left(\sum_{k=n}^{\infty} \lambda_k x_k \right)^d \leq \left(\frac{d}{c-1} \right)^d \sum_{n=1}^{\infty} \lambda_n (\Lambda_n^*)^{d-c} x_n^d.$$

The constant is best possible.

The above lemma is Corollary 6 to Theorem 2 of [2] and only the special case $d = c$ is needed for the proof of Theorem 1.3.

Lemma 4.2. *Let* $p < 0$. *Let integers* $M \geq N \geq 1$ *be fixed. For any positive sequence* $(a_n)_{n=1}^M$ *with* $A_{n,M} = \sum_{k=n}^{M} a_k$, *we have*

$$\sum_{k=1}^{N} a_k A_{k,M}^{p-1} \leq \left(1 - \frac{1}{p} \right) A_{N,M}^p.$$

Competing interests

The author declares that he has no competing interests.

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