VARIANTS OF ČEBYŠEV'S INEQUALITY WITH APPLICATIONS

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Several variants of Čebyšev's inequality for two monotonic n-tuples and also $k \geq 3$ non-negative n-tuples monotonic in the same direction are presented. Immediately after that their refinements of Ostrowski's type are given. Obtained results are used to prove generalizations of discrete Milne's inequality and its converse in which weights satisfy conditions as in the Jensen-Steffensen inequality.

1. Introduction

In 2003 Mercer gave the following interesting variant of the discrete Jensen's inequality (see, e.g., [8, page 43]) for convex functions.

Theorem 1.1 [4, Theorem 1]. *If f is a convex function on an interval containing n-tuple $\mathbf{x} = (x_1, \ldots, x_n)$ such that $0 < x_1 \leq x_2 \leq \cdots \leq x_n$ and $\mathbf{w} = (w_1, \ldots, w_n)$ is positive n-tuple with $\sum_{i=1}^{n} w_i = 1$, then*

$$f\left(x_1 + x_n - \sum_{i=1}^{n} w_i x_i\right) \leq f(x_1) + f(x_n) - \sum_{i=1}^{n} w_i f(x_i). \tag{1.1}$$

Two years later his result was generalized as it is stated below.

Theorem 1.2 [1, Theorem 2]. *Let $[a,b]$ be an interval in $\mathbb{R}$, $a < b$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a monotonic n-tuple in $[a,b]^n$, and let $\mathbf{w} = (w_1, \ldots, w_n)$ be a real n-tuple such that*

$$0 \leq W_k \leq W_n (k = 1, \ldots, n-1), \quad W_n > 0, \tag{1.2}$$

where $W_k = \sum_{i=1}^{k} w_i (k = 1, \ldots, n)$. *If function* $f : [a,b] \to \mathbb{R}$ *is convex, then*

$$f\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right) \le f(a) + f(b) - \frac{1}{W_n}\sum_{i=1}^{n} w_i f(x_i). \tag{1.3}$$

As we can see, here the condition $w_i > 0$ $(i = 1, \ldots, n)$ is relaxed on the conditions (1.2) as in the well-known Jensen-Steffensen inequality for sums (see, e.g, [8, page 57]).

Remark 1.3. It can be easily proved that for a real n-tuple $\mathbf{w}$ which satisfies (1.2) and for any monotonic n-tuple $\mathbf{x} \in [a,b]^n$ the inequalities

$$a \le \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \le b, \tag{1.4}$$

hold. From (1.4) we can also conclude that $a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i \in [a,b]$.

In this paper we present "Mercer's type" variants of several well-known inequalities.

In Section 2 we give generalizations of the discrete Čebyšev's inequality for two monotonic n-tuples and also for $k \ge 3$ nonnegative n-tuples monotonic in the same direction, in which weights $\mathbf{w}$ satisfy the conditions (1.2). Immediately after Mercer's type variants of those inequalities are presented. In Section 3 we give analogous variants of Pečarić's generalizations of the discrete Ostrowski's inequalities. In Section 4 we use results from Section 2 to obtain generalizations of Milne's inequality and its converse. Mercer's type variants of Milne's inequality and its converse are also given.

2. Variants of Čebyšev's inequality

A classic result due to Čebyšev (1882, 1883) is stated as follows. Let $\mathbf{w}$ be a nonnegative n-tuple. If real n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ are monotonic in the same direction, then

$$\sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i \le \sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i. \tag{2.1}$$

If $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite directions, the inequality (2.1) is reversed.

Although the proof of the following generalization of the inequality (2.1) has been already known (see [6]) for the sake of clarity, we will briefly present it here.

THEOREM 2.1. *Let* $\mathbf{w} = (w_1, \ldots, w_n)$ *be a real n-tuple such that (1.2) is satisfied. Then for any real n-tuples* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ *monotonic in the same direction the inequality (2.1) holds. If $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite directions, (2.1) is reversed.*

Proof. Using the well-known Abel's identity it can be proved that the following identity holds:

$$\sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i$$
$$= \sum_{k=1}^{n-1}\left[\sum_{l=1}^{k-1}\overline{W}_{k+1} W_l (x_{l+1} - x_l)(y_{k+1} - y_k) + \sum_{l=k+1}^{n}\overline{W}_l W_k (x_l - x_{l-1})(y_{k+1} - y_k)\right], \tag{2.2}$$

where $\overline{W}_k = \sum_{i=k}^{n} w_i$. Suppose that $\mathbf{x}$ and $\mathbf{y}$ are monotonic in the same direction. Then

$$(x_{i+1} - x_i)(y_{j+1} - y_j) \geq 0 \tag{2.3}$$

for all $i, j \in \{1, \ldots, n-1\}$. Furthermore, the conditions (1.2) on n-tuple $\mathbf{w}$ imply that also

$$\overline{W}_k \geq 0 \quad (k = 1, \ldots, n), \tag{2.4}$$

so from identity (2.2) we may conclude that

$$\sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i \geq 0. \tag{2.5}$$

If $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite directions, we have

$$(x_{i+1} - x_i)(y_{j+1} - y_j) \leq 0 \tag{2.6}$$

for all $i, j \in \{1, \ldots, n-1\}$, so the reverse of (2.1) immediately follows.

This completes the proof. $\qquad \square$

In the next theorem we give a Mercer's type variant of the inequality (2.1).

THEOREM 2.2. *Let $n \geq 2$ and let $\mathbf{w}$ be a real n-tuple such that (1.2) is satisfied. Let $[a,b]$ and $[c,d]$ be intervals in $\mathbb{R}$, where $a < b$, $c < d$. Then for any real n-tuples $\mathbf{x} \in [a,b]^n$ and $\mathbf{y} \in [c,d]^n$ monotonic in the same direction,*

$$\left(a + b - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i\right)\left(c + d - \frac{1}{W_n}\sum_{i=1}^{n} w_i y_i\right) \leq ac + bd - \frac{1}{W_n}\sum_{i=1}^{n} w_i x_i y_i. \tag{2.7}$$

If $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite directions, the inequality (2.7) is reversed.

Proof. Without any loss of generality we may suppose that n-tuples $\mathbf{x}$ and $\mathbf{y}$ are both monotonically decreasing (in other cases the proof is similar). We define $(n+2)$-tuples $\mathbf{w}' = (w_1', \ldots, w_{n+2}')$, $\mathbf{x}' = (x_1', \ldots, x_{n+2}')$, and $\mathbf{y}' = (y_1', \ldots, y_{n+2}')$ as

$$\begin{aligned}
w_1' &= 1, & w_2' &= -\frac{w_1}{W_n}, \ldots, w_{n+1}' = -\frac{w_n}{W_n}, & w_{n+2}' &= 1, \\
x_1' &= b, & x_2' &= x_1, \ldots, x_{n+1}' = x_n, & x_{n+2}' &= a, \\
y_1' &= d, & y_2' &= y_1, \ldots, y_{n+1}' = y_n, & y_{n+2}' &= c.
\end{aligned} \tag{2.8}$$

Obviously, $\mathbf{x}'$ and $\mathbf{y}'$ are both monotonically decreasing and we have

$$0 \leq W_k' \leq 1 \quad (k = 1, \ldots, n+1), \qquad W_{n+2}' = 1, \tag{2.9}$$

so we may apply Theorem 2.1 on $(n+2)$-tuples $\mathbf{w}'$, $\mathbf{x}'$, and $\mathbf{y}'$ to obtain

$$\sum_{i=1}^{n+2} w_i' x_i' \sum_{i=1}^{n+2} w_i' y_i' \leq \sum_{i=1}^{n+2} w_i' \sum_{i=1}^{n+2} w_i' x_i' y_i' \tag{2.10}$$

from which we can easily get (2.7). $\qquad \square$

Čebyšev's inequality can be generalized for $k \geq 3$ nonnegative n-tuples monotonic in the same direction with nonnegative weights $\mathbf{w}$ (see, e.g., [8, page 198]). Here we give an analogous generalization of Čebyšev's inequality for $k \geq 3$ nonnegative n-tuples in which weights $\mathbf{w}$ satisfy the conditions (1.2). Partial order "$\leq$" on $\mathbb{R}^k$ here is defined as

$$(x_1,\ldots,x_k) \leq (y_1,\ldots,y_k) \iff x_1 \leq y_1 \wedge \cdots \wedge x_k \leq y_k. \tag{2.11}$$

In order to simplify our results, we will consider only weights $\mathbf{w}$ with sum 1.

THEOREM 2.3. *Let $n \geq 2$ and let $\mathbf{w}$ be a real n-tuple such that*

$$0 \leq W_k \leq 1 \quad (k = 1,\ldots,n-1), \qquad W_n = 1. \tag{2.12}$$

Let $k \geq 2$ and let $I \subseteq [0,+\infty)^k$. Then for any $\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)} \in I$ such that

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}, \tag{2.13}$$

the following holds:

$$\prod_{i=1}^{k} \sum_{j=1}^{n} w_j x_i^{(j)} \leq \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)}. \tag{2.14}$$

Proof. The proof of (2.14) is by induction on k. The case $k = 2$ follows from Theorem 2.1. Suppose that (2.14) is valid for all l, $2 \leq l \leq k$. We have

$$\sum_{j=1}^{n} w_j \prod_{i=1}^{k+1} x_i^{(j)} = \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} x_{k+1}^{(j)}, \tag{2.15}$$

and we know that

$$\prod_{i=1}^{k} \sum_{j=1}^{n} w_j x_i^{(j)} \geq 0, \qquad \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} \geq 0, \qquad \sum_{j=1}^{n} w_j x_{k+1}^{(j)} \geq 0 \tag{2.16}$$

(see Remark 1.3). We define nonnegative n-tuple $\mathbf{y}$ as

$$y_j = \prod_{i=1}^{k} x_i^{(j)} \quad (j = 1,\ldots,n). \tag{2.17}$$

It can be easily seen that $\mathbf{y}$ is monotonic in the same sense as $(\mathbf{x}^{(1)},\ldots,\mathbf{x}^{(n)})$, that is, $\mathbf{y}$ is monotonic in the same sense as $(x_{k+1}^{(1)},\ldots,x_{k+1}^{(n)})$, so we may apply (2.1) and our induction hypothesis in (2.15) to obtain

$$\sum_{j=1}^{n} w_j \prod_{i=1}^{k+1} x_i^{(j)} = \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} x_{k+1}^{(j)} \geq \left(\sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)} \right) \left(\sum_{j=1}^{n} w_j x_{k+1}^{(j)} \right)$$
$$\geq \left(\prod_{i=1}^{k} \sum_{j=1}^{n} w_j x_i^{(j)} \right) \left(\sum_{j=1}^{n} w_j x_{k+1}^{(j)} \right) = \prod_{i=1}^{k+1} \sum_{j=1}^{n} w_j x_i^{(j)}, \tag{2.18}$$

so by induction the result holds. $\qquad\square$

In the next theorem we give a Mercer's type variant of (2.14).

Theorem 2.4. *Let $n \geq 2$ and let $\mathbf{w}$ be a real n-tuple such that (2.12) is satisfied. Let $k \geq 2$ and let $I = [a_1, b_1] \times \cdots \times [a_k, b_k] \subset [0, +\infty)^k$. Then for any $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(n)} \in I$ such that*

$$\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)} \quad or \quad \mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}, \tag{2.19}$$

the following holds:

$$\prod_{i=1}^{k} \left(a_i + b_i - \sum_{j=1}^{n} w_j x_i^{(j)} \right) \leq \prod_{i=1}^{k} a_i + \prod_{i=1}^{k} b_i - \sum_{j=1}^{n} w_j \prod_{i=1}^{k} x_i^{(j)}. \tag{2.20}$$

Proof. Suppose that $\mathbf{x}^{(1)} \leq \cdots \leq \mathbf{x}^{(n)}$. We define vectors $\boldsymbol{\xi}^{(j)} \in [0, +\infty)^k$ ($j = 1, \ldots, n+2$) and weights $\mathbf{w}'$ as

$$\xi_i^{(1)} = a_i, \quad \xi_i^{(n+2)} = b_i \quad (i = 1, \ldots, k),$$

$$\boldsymbol{\xi}^{(j)} = \boldsymbol{x}^{(j-1)} \quad (j = 2, \ldots, n+1), \tag{2.21}$$

$$w_1' = 1, \qquad w_2' = -w_1, \ldots, w_{n+1}' = w_n, \qquad w_{n+2}' = 1.$$

Obviously, we have $\boldsymbol{\xi}^{(1)} \leq \cdots \leq \boldsymbol{\xi}^{(n+2)}$ and

$$0 \leq W_k' \leq 1 \quad (k = 1, \ldots, n+1), \qquad W_{n+2}' = 1. \tag{2.22}$$

We can apply Theorem 2.3 on $\boldsymbol{\xi}^{(j)}$ ($j = 1, \ldots, n+2$) and $\mathbf{w}'$ to obtain

$$\prod_{i=1}^{k} \sum_{j=1}^{n+2} w_j' \xi_i^{(j)} \leq \sum_{j=1}^{n+2} w_j'^{n+2} w_j' \prod_{i=1}^{k} \xi_i^{(j)}, \tag{2.23}$$

from which (2.20) immediately follows. If $\mathbf{x}^{(1)} \geq \cdots \geq \mathbf{x}^{(n)}$, the proof is similar. $\square$

3. Variants of Pečarić's inequalities

In 1984 Pečarić proved several generalizations of the discrete Ostrowski's inequalities. Here we give two of them which are interesting to us because they are refinements of Theorem 2.1.

Theorem 3.1 [7, Theorem 3]. *Let $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ be real n-tuples monotonic in the same direction and let $\mathbf{w} = (w_1, \ldots, w_n)$ be a real n-tuple such that*

$$0 \leq W_k \leq W_n \quad (k = 1, \ldots, n-1). \tag{3.1}$$

If m and r are nonnegative real numbers such that

$$|x_{k+1} - x_k| \geq m, \quad |y_{k+1} - y_k| \geq r \quad (k = 1, \ldots, n-1), \tag{3.2}$$

then

$$T(\mathbf{x}, \mathbf{y}; \mathbf{w}) \geq mr T(\mathbf{e}, \mathbf{e}; \mathbf{w}) \geq 0, \tag{3.3}$$

where

$$T(\mathbf{x}, \mathbf{y}; \mathbf{w}) = \sum_{i=1}^{n} w_i \sum_{i=1}^{n} w_i x_i y_i - \sum_{i=1}^{n} w_i x_i \sum_{i=1}^{n} w_i y_i,$$
$$\mathbf{e} = (0, 1, \ldots, n-1).$$

(3.4)

If $\mathbf{x}$ *and* $\mathbf{y}$ *are monotonic in opposite directions, then*

$$T(\mathbf{x}, \mathbf{y}; \mathbf{w}) \le -mr T(\mathbf{e}, \mathbf{e}; \mathbf{w}) \le 0.$$

(3.5)

THEOREM 3.2 [7, Theorem 4]. *Let* $\mathbf{x}$ *and* $\mathbf{y}$ *be real n-tuples such that*

$$\left| x_{k+1} - x_k \right| \le M, \quad \left| y_{k+1} - y_k \right| \le R \quad (k = 1, \ldots, n-1)$$

(3.6)

hold for some nonnegative real numbers M and R, and let $\mathbf{w}$ *be a real n-tuple such that (3.1) is valid. Then*

$$\left| T(\mathbf{x}, \mathbf{y}; \mathbf{w}) \right| \le MR T(\mathbf{e}, \mathbf{e}; \mathbf{w}).$$

(3.7)

In the next two theorems we give Mercer's type variants of Theorems 3.1 and 3.2 which are refinements of Theorem 2.2.

THEOREM 3.3. *Let* $n \ge 2$ *and let* $\mathbf{w}$ *be a real n-tuple such that (2.12) is valid. Let* $[a, b]$, $[c, d]$ *be intervals in* $\mathbb{R}$, *where* $a < b$, $c < d$. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in [a, b]^n$ *and* $\mathbf{y} = (y_1, \ldots, y_n) \in [c, d]^n$ *be monotonic n-tuples, and let m and r be nonnegative real numbers such that*

$$\begin{aligned}
&\min_{1 \le i \le n} x_i - a \ge m, \qquad b - \max_{1 \le i \le n} x_i \ge m, \\
&\left| x_{k+1} - x_k \right| \ge m \quad (k = 1, \ldots, n-1), \\
&\min_{1 \le i \le n} y_i - c \ge r, \qquad d - \max_{1 \le i \le n} y_i \ge r, \\
&\left| y_{k+1} - y_k \right| \ge r \quad (k = 1, \ldots, n-1).
\end{aligned}$$

(3.8)

If $\mathbf{x}$ *and* $\mathbf{y}$ *are monotonic in the same direction, then*

$$\widetilde{T}(\mathbf{x}, \mathbf{y}; \mathbf{w}) \ge mr \left[\widetilde{T}(\mathbf{f}, \mathbf{f}; \mathbf{w}) + 2n \right] \ge 0,$$

(3.9)

where

$$\widetilde{T}(\mathbf{x}, \mathbf{y}; \mathbf{w}) = ac + bd - \sum_{i=1}^{n} w_i x_i y_i - \left(a + b - \sum_{i=1}^{n} w_i x_i \right) \left(c + d - \sum_{i=1}^{n} w_i y_i \right),$$
$$\mathbf{f} = (1, \ldots, n) \in [1, n]^n.$$

(3.10)

If $\mathbf{x}$ *and* $\mathbf{y}$ *are monotonic in opposite directions, then*

$$\widetilde{T}(\mathbf{x}, \mathbf{y}; \mathbf{w}) \le -mr \left[\widetilde{T}(\mathbf{f}, \mathbf{f}; \mathbf{w}) + 2n \right] \le 0.$$

(3.11)

Proof. Suppose that n-tuples $\mathbf{x}$ and $\mathbf{y}$ are both monotonically decreasing (if $\mathbf{x}$ and $\mathbf{y}$ are monotonically increasing, the proof is similar). We define $(n+2)$-tuples $\mathbf{w}' = (w_1',\ldots,$ $w_{n+2}')$, $\mathbf{x}' = (x_1',\ldots,x_{n+2}')$, and $\mathbf{y}' = (y_1',\ldots,y_{n+2}')$ as

$$
\begin{aligned}
w_1' &= 1, & w_2' &= -w_1,\ldots,w_{n+1}' = -w_n, & w_{n+2}' &= 1, \\
x_1' &= b, & x_2' &= x_1,\ldots,x_{n+1}' = x_n, & x_{n+2}' &= a, \\
y_1' &= d, & y_2' &= y_1,\ldots,y_{n+1}' = y_n, & y_{n+2}' &= c.
\end{aligned}
\tag{3.12}
$$

Obviously, $\mathbf{x}'$ and $\mathbf{y}'$ are both monotonically decreasing and we have

$$
\begin{aligned}
0 \le W_k' \le 1 \quad (k = 1,\ldots,n+1), &\qquad W_{n+2}' = 1, \\
|x_{k+1}' - x_k'| \ge m, \quad |y_{k+1}' - y_k'| \ge r &\quad (k = 1,\ldots,n+1).
\end{aligned}
\tag{3.13}
$$

From Theorem 3.1 we have

$$
T(\mathbf{x}',\mathbf{y}';\mathbf{w}') \ge mr\,T(\mathbf{e}',\mathbf{e}';\mathbf{w}') \ge 0,
\tag{3.14}
$$

where

$$
\mathbf{e}' = (0,1,\ldots,n+1).
\tag{3.15}
$$

From that we immediately obtain

$$
\begin{aligned}
ac + bd - \sum_{i=1}^{n} w_i x_i y_i &- \left(a + b - \sum_{i=1}^{n} w_i x_i\right)\left(c + d - \sum_{i=1}^{n} w_i y_i\right) \\
&\ge mr\left[\sum_{i=1}^{n+2} w_i'(i-1)^2 - \left(\sum_{i=1}^{n+2} w_i'(i-1)\right)^2\right] \\
&= mr\left[(n+1)^2 - \sum_{i=1}^{n} w_i i^2 - \left(n+1 - \sum_{i=1}^{n} w_i i\right)^2\right] \ge 0,
\end{aligned}
\tag{3.16}
$$

that is,

$$
\widetilde{T}(\mathbf{x},\mathbf{y};\mathbf{w}) \ge mr\left[\widetilde{T}(\mathbf{f},\mathbf{f};\mathbf{w}) + 2n\right] \ge 0.
\tag{3.17}
$$

If n-tuples $\mathbf{x}$ and $\mathbf{y}$ are monotonic in opposite directions, the proof is similar. $\qquad\square$

Theorem 3.4. *Let $n \ge 2$ and let $\mathbf{w}$ be a real n-tuple such that (2.12) is valid. Let $[a,b]$, $[c,d]$ be intervals in $\mathbb{R}$, where $a < b$, $c < d$. Let $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$, $\mathbf{y} = (y_1,\ldots,y_n) \in [c,d]^n$ and let M and R be nonnegative real numbers such that*

$$
\begin{aligned}
|x_1 - a| &\le M, & |b - x_n| &\le M, \\
|x_{k+1} - x_k| &\le M &(k = 1,\ldots,n-1), \\
|y_1 - c| &\le R, & |d - y_n| &\le R, \\
|y_{k+1} - y_k| &\le R &(k = 1,\ldots,n-1).
\end{aligned}
\tag{3.18}
$$

Then

$$\left|\widetilde{T}(\mathbf{x},\mathbf{y};\mathbf{w})\right| \le MR[\widetilde{T}(\mathbf{f},\mathbf{f};\mathbf{w}) + 2n] \le 0. \tag{3.19}$$

Proof. Similarly as in Theorem 3.3.   □

COROLLARY 3.5. *Let $n \ge 2$ and let $[a,b]$ be an interval in $\mathbb{R}$ where $a < b$. Then for all $\mathbf{x} = (x_1,\ldots,x_n) \in [a,b]^n$,*

$$\left[na^2 + nb^2 - \sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(na + nb - \sum_{i=1}^{n} x_i\right)^2\right] \frac{12}{n(n+1)(5n+7)} \ge m^2, \tag{3.20}$$

where

$$m = \min_{0 \le i < j \le n+1} |x_i - x_j|, \qquad x_0 = a, \qquad x_{n+1} = b. \tag{3.21}$$

Proof. Directly from Theorem 3.3.   □

COROLLARY 3.6. *Let $\mathbf{x} = (x_1,\ldots,x_n)$, $\mathbf{y} = (y_1,\ldots,y_n)$, M and R be defined as in Theorem 3.4. Then*

$$\left|nac + nbd - \sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(na + nb - \sum_{i=1}^{n} x_i\right)\left(nc + nd - \sum_{i=1}^{n} y_i\right)\right| \le \frac{n(n+1)(5n+7)}{12} MR. \tag{3.22}$$

Proof. Directly from Theorem 3.4.   □

The above results are variants of some Lupaş' results [3].

4. Applications: inequality of Milne and its converse

In 1925 Milne [5] obtained the following interesting integral inequality for positive functions f and g which are integrable on $[a,b]$:

$$\int_a^b \frac{f(x)g(x)}{f(x)+g(x)} dx \int_a^b [f(x) + g(x)] dx \le \int_a^b f(x)dx \int_a^b g(x)dx. \tag{4.1}$$

In 2000 Rao [9] combined Milne's inequality and the well-known inequality between arithmetic and geometric means to obtain the following double inequality for sums.

PROPOSITION 4.1. *Let $n \ge 2$ and let $w_i > 0$ ($i = 1,2,\ldots,n$) be real numbers with $\sum_{i=1}^{n} w_i = 1$. Then for all real numbers $p_i \in \langle -1,1 \rangle$ ($i = 1,\ldots,n$),*

$$\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \le \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i}\right)\left(\sum_{i=1}^{n} \frac{w_i}{1 + p_i}\right) \le \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2}\right)^2. \tag{4.2}$$

Two years later Alzer and Kovačec obtained the following refinement of (4.2).

THEOREM 4.2 [2, Theorem 1]. *Let $n \geq 2$ and let $w_i > 0$ $(i = 1, 2, \ldots, n)$ be real numbers with $\sum_{i=1}^{n} w_i = 1$. Then for all real numbers $p_i \in [0, 1\rangle$ $(i = 1, \ldots, n)$,*

$$\left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \right)^{\alpha} \leq \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i} \right) \left(\sum_{i=1}^{n} \frac{w_i}{1 + p_i} \right) \leq \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \right)^{\beta} \tag{4.3}$$

with the best possible exponents

$$\alpha = 1, \qquad \beta = 2 - \min_{1 \leq i \leq n} w_i. \tag{4.4}$$

We note here that the crucial step in the proof of Theorem 4.2 was performed by using a discrete variant of the Čebyšev's inequality (see, e.g., [8, page 197]) which itself was generalized in Section 2. This enables us to give the following generalization of Theorem 4.2.

THEOREM 4.3. *Let $n \geq 2$ and let $\mathbf{w} = (w_1, \ldots, w_n)$ be a real n-tuple such that (2.12) is satisfied. Then for all $\alpha \in \langle -\infty, 1]$, $\beta \in [2 - \min_{1 \leq i \leq n} W_i, +\infty\rangle$ and for all monotonic n-tuples $\mathbf{p} = (p_1, \ldots, p_n) \in [0, 1\rangle^n$,*

$$\left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \right)^{\alpha} \leq \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i} \right) \left(\sum_{i=1}^{n} \frac{w_i}{1 + p_i} \right) \leq \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \right)^{\beta} \tag{4.5}$$

with the best possible exponents

$$\alpha = 1, \qquad \beta = 2 - \min_{1 \leq i \leq n} W_i. \tag{4.6}$$

Proof. We follow the idea of the proof given in [2]. Suppose that $1 > p_1 \geq p_2 \geq \cdots \geq p_n \geq 0$. It can be easily seen that

$$0 < \frac{1}{1 + p_1} \leq \frac{1}{1 + p_2} \leq \cdots \leq \frac{1}{1 + p_n} \leq 1,$$

$$\frac{1}{1 - p_1} \geq \frac{1}{1 - p_2} \geq \cdots \geq \frac{1}{1 - p_n} \geq 1, \tag{4.7}$$

$$\frac{1}{1 - p_1^2} \geq \frac{1}{1 - p_2^2} \geq \cdots \geq \frac{1}{1 - p_n^2} \geq 1,$$

so in this case (see Remark 1.3) we know that

$$\sum_{i=1}^{n} \frac{w_i}{1 - p_i} \geq 1, \qquad \sum_{i=1}^{n} \frac{w_i}{1 + p_i} > 0, \qquad \sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \geq 1. \tag{4.8}$$

Let $w = \min_{1 \leq i \leq n} W_i$. We define function $f : [0, 1\rangle^n \to \mathbb{R}$ as

$$f(p_1, \ldots, p_n) = (2 - w) \log \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i^2} \right) - \log \left(\sum_{i=1}^{n} \frac{w_i}{1 - p_i} \right) - \log \left(\sum_{i=1}^{n} \frac{w_i}{1 + p_i} \right). \tag{4.9}$$

For fixed $k \in \{1, \ldots, n - 1\}$ we define function $f_k : [0, 1\rangle \to \mathbb{R}$ as

$$f_k(p) = f(p, \ldots, p, p_{k+1}, \ldots, p_n). \tag{4.10}$$

Let $p \in [p_{k+1}, 1\rangle$. We have

$$f'_k(p) = \frac{W_k D}{(1-p^2)ABC},$$ (4.11)

where

$$A = W_k + \sum_{i=k+1}^{n} w_i \frac{1-p^2}{1-p_i^2}, \qquad B = W_k + \sum_{i=k+1}^{n} w_i \frac{1-p}{1-p_i}, \qquad C = W_k + \sum_{i=k+1}^{n} w_i \frac{1+p}{1+p_i},$$ (4.12)

$$D = A[(1-p)B - (1+p)C] + 2(2-w)pBC.$$ (4.13)

We define n-tuples $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ with

$$x_i = 1, \quad y_i = 1 \quad (i = 1, \ldots, k),$$
$$x_i = \frac{1-p}{1-p_i}, \quad y_i = \frac{1+p}{1+p_i} \quad (i = k+1, \ldots, n),$$ (4.14)

which are obviously monotonic in opposite directions. From Theorem 2.1 we have

$$\left(\sum_{i=1}^{n} w_i x_i \right) \left(\sum_{i=1}^{n} w_i y_i \right) \geq \sum_{i=1}^{n} w_i x_i y_i,$$ (4.15)

that is, $BC \geq A$, and from Remark 1.3 we know that A, B, and C are all positive. This enables us to conclude that

$$\frac{D}{A} \geq (1-p)B - (1+p)C + 2(2-w)p$$
$$= 2p(2 - w - W_k) + \sum_{i=k+1}^{n} w_i \left(\frac{(1-p)^2}{1-p_i} - \frac{(1+p)^2}{1+p_i} \right).$$ (4.16)

It can be easily seen that

$$-4p = (1-p)^2 - (1+p)^2 \leq \frac{(1-p)^2}{1-p_{k+1}} - \frac{(1+p)^2}{1+p_{k+1}}$$
$$\leq \cdots \leq \frac{(1-p)^2}{1-p_n} - \frac{(1+p)^2}{1+p_n},$$ (4.17)

so we have

$$\sum_{i=1}^{k} w_i(-4p) + \sum_{i=k+1}^{n} w_i \left(\frac{(1-p)^2}{1-p_i} - \frac{(1+p)^2}{1+p_i} \right) \geq -4p,$$ (4.18)

that is,

$$\sum_{i=k+1}^{n} w_i \left(\frac{(1-p)^2}{1-p_i} - \frac{(1+p)^2}{1+p_i} \right) \geq -4p + 4pW_k.$$ (4.19)

From (4.19) and (4.16) we obtain

$$\frac{D}{A} \geq 2p(W_k - w) \geq 0, \tag{4.20}$$

which implies that the function f_k is increasing on $[p_{k+1}, 1\rangle$. Using that fact we obtain

$$f(p_1,\ldots,p_n) = f_1(p_1) \geq f_1(p_2) = f_2(p_2) \geq f_2(p_3) = f_3(p_3)$$
$$\geq \cdots \geq f_{n-1}(p_n) = -(1-w)\log(1-p_n^2) \geq 0, \tag{4.21}$$

which implies

$$\left(\sum_{i=1}^{n} \frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n} \frac{w_i}{1+p_i}\right) \leq \left(\sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{2-w}, \tag{4.22}$$

that is, the right inequality in (4.5) holds for $\beta = 2 - \min_{1 \leq i \leq n} W_i$. Since

$$\sum_{i=1}^{n} \frac{w_i}{1-p_i^2} \geq 1, \tag{4.23}$$

it is clear that it also holds for all $\beta \geq 2 - \min_{1 \leq i \leq n} W_i$.

A similar argument as in [2] shows that $\beta = 2 - \min_{1 \leq i \leq n} W_i$ gives the best upper bound in (4.5): if $W_k = \min_{1 \leq i \leq n} W_i$, we simply choose n-tuple $\mathbf{p} = (p_1, \ldots, p_n)$ defined as

$$p_1 = \cdots = p_k = q, \quad p_{k+1} = \cdots = p_n = 0, \quad q \in \langle 0, 1 \rangle, \tag{4.24}$$

and for such $\mathbf{p}$ and $\mathbf{w}$ we obtain that β must satisfy the condition $\beta \geq 2 - W_k$.

The left-hand side of (4.5) is a simple consequence of Theorem 2.1. If we define

$$x_i = \frac{1}{1-p_i}, \quad y_i = \frac{1}{1+p_i} \quad (i = 1,\ldots,n), \tag{4.25}$$

then n-tuples $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$ are monotonic in opposite directions, so we have

$$\sum_{i=1}^{n} \frac{w_i}{1-p_i^2} \leq \left(\sum_{i=1}^{n} \frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n} \frac{w_i}{1+p_i}\right). \tag{4.26}$$

Furthermore, (4.23) implies

$$\left(\sum_{i=1}^{n} \frac{w_i}{1-p_i^2}\right)^{\alpha} \leq \left(\sum_{i=1}^{n} \frac{w_i}{1-p_i}\right)\left(\sum_{i=1}^{n} \frac{w_i}{1+p_i}\right) \tag{4.27}$$

for all $\alpha \leq 1$.

The same argument as in [2] shows that $\alpha = 1$ gives the best lower bound for (4.5). In case $0 \le p_1 \le \cdots \le p_n < 1$ the proof is similar. $\qquad\square$

In the next theorem we give a Mercer's type variant of (4.5).

Theorem 4.4. *Let $n \ge 2$ and let $\mathbf{w} = (w_1, \dots, w_n)$ be a real n-tuple such that (2.12) is satisfied. Then for all $\alpha \in \langle -\infty, 1]$, $\beta \in [2, +\infty\rangle$ and for all monotonic n-tuples $\mathbf{p} = (p_1, \dots, p_n) \in [p,q]^n$, where $[p,q] \subseteq [0,1\rangle$ and $p < q$,*

$$
\left(\frac{1}{1-p^2} + \frac{1}{1-q^2} - \sum_{i=1}^{n} \frac{w_i}{1-p_i^2} \right)^{\alpha}
$$

$$
\le \left(\frac{1}{1-p} + \frac{1}{1-q} - \sum_{i=1}^{n} \frac{w_i}{1-p_i} \right) \left(\frac{1}{1+p} + \frac{1}{1+q} - \sum_{i=1}^{n} \frac{w_i}{1+p_i} \right) \tag{4.28}
$$

$$
\le \left(\frac{1}{1-p^2} + \frac{1}{1-q^2} - \sum_{i=1}^{n} \frac{w_i}{1-p_i^2} \right)^{\beta},
$$

with the best possible exponents

$$
\alpha = 1, \qquad \beta = 2. \tag{4.29}
$$

Proof. Suppose that $q \ge p_1 \ge p_2 \ge \cdots \ge p_n \ge p$. We define $(n+2)$-tuples $\mathbf{w}' = (w_1', \dots, w_{n+2}')$ and $\mathbf{p}' = (p_1', \dots, p_{n+2}') \in [0,1\rangle^n$ with

$$
\begin{aligned}
w_1' = 1, \qquad & w_2' = -w_1, \dots, w_{n+1}' = -w_n, \qquad w_{n+2}' = 1, \\
p_1' = q, \qquad & p_2' = p_1, \dots, p_{n+1}' = p_n, \qquad p_{n+2}' = p.
\end{aligned} \tag{4.30}
$$

We have

$$
0 \le W_k' \le 1 \quad (k = 1, \dots, n+1), \qquad W_{n+2}' = 1, \qquad \min_{1 \le i \le n} W_i' = 0. \tag{4.31}
$$

From Remark 1.3 we know that

$$
\sum_{i=1}^{n+2} \frac{w_i'}{1 - p_i'^2} \ge 1, \tag{4.32}
$$

so the left side and the right side of (4.28) are well defined. If we apply Theorem 4.3 on $(n+2)$-tuples $\mathbf{w}'$ and $\mathbf{p}'$, we obtain

$$
\left(\sum_{i=1}^{n+2} \frac{w_i'}{1 - p_i'^2} \right)^{\alpha} \le \left(\sum_{i=1}^{n+2} \frac{w_i'}{1 - p_i'} \right) \left(\sum_{i=1}^{n+2} \frac{w_i'}{1 + p_i'} \right) \le \left(\sum_{i=1}^{n+2} \frac{w_i'}{1 - p_i'^2} \right)^{\beta}, \tag{4.33}
$$

from which (4.28) immediately follows.

If $p \le p_1 \le \cdots \le p_n \le q$, the proof is similar. $\qquad\square$

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