EXISTENCE AND APPROXIMATE SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

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Received 10 October 2003 and in revised form 4 December 2003

We investigate the existence of continuous solutions on compact intervals of some nonlinear integral equations. The existence of such solutions is based on some well-known fixed point theorems in Banach spaces such as Schaefer fixed point theorem, Schauder fixed point theorem, and Leray-Schauder principle. A special interest is devoted to the study of nonlinear Volterra equations and to the numerical treatment of these equations.

1. Introduction

In the first part of this work, we study the existence of a solution of the following functional integral equation:

$$x(t) = f(t) + \int_a^b K(t,s)x(s)\,ds + \int_a^b V(t,s)g\big(s,x(s)\big)\,ds, \quad -\infty < a \le t \le b < +\infty. \tag{1.1}$$

Note that the previous integral equation can be considered as a nonlinear Fredholm equation expressed as a perturbed linear equation. A Krasnoselkii-Schafer fixed point theorem [4] is used to prove the existence of a solution of some special cases of (1.1), see [8]. The general nonlinear integral equation has the following form:

$$x(t) = f(t) + \int_a^b g\big(t,s,x(s)\big)\,ds, \quad -\infty \le a \le t \le b \le +\infty. \tag{1.2}$$

We should mention that an extensive work has been done in the study of the solutions of various types of (1.2), see, for example, [1, 2, 5, 7, 11, 13, 15, 16, 17, 19]. Usually the existence of a solution of (1.2) starts with some conditions on the function $g(t,s,x)$ as well as the integration bounds a, b and the function $f(\cdot)$. Based on these conditions, a Banach space is chosen in such a way that the existence problem is converted to a fixed point problem of an operator over this Banach space.

To prove the existence of a continuous solution of the integral equation (1.1), we use some conditions on the function $f(\cdot)$, the kernels $K(t,s)$, $V(t,s)$ as well as on the function $g(t,x)$. By using these conditions, we define a completely continuous operator T over

Journal of Inequalities and Applications 2005:5 (2005) 569–581
DOI: 10.1155/JIA.2005.569

the Banach space $C([a,b])$ whose fixed points are solutions of (1.1). The well-known fixed point theorem of Schaefer [20] is used to prove the existence of a fixed point of the operator T. Also, by introducing a convenient new norm $\|\cdot\|_\mu$ on the space $C([a,b])$, we study the existence of continuous solutions of the general nonlinear equation (1.2) with finite bounds a and b.

In the second part of this work, we study the existence of continuous solution of the following nonlinear Volterra equation:

$$x(t) = f(t) + \int_a^t g(t,s,x(s))\,ds = f(t) + Tx(t), \quad -\infty < a \le t \le b < \infty, \tag{1.3}$$

where $f(\cdot) \in C([a,b])$. The main tool in the proof of the existence of a solution of (1.3) is the Leray-Schauder principle combined with a general version of Gronwall's inequality. Moreover, we prove the uniqueness of the solution of (1.3) by showing that there exists $n \in N$ such that T^n is a contraction on some closed ball containing all possible continuous solutions of (1.3).

This paper is organized as follows. In Section 2, we prove the existence of the solutions of some special cases of (1.1) and (1.2). In Section 3, we investigate the existence and the uniqueness of a solution of the nonlinear Volterra equation (1.3). Finally in Section 4, we provide the reader with a numerical scheme for solving nonlinear Volterra equations.

2. Existence of a solution of nonlinear integral equations

In the first part of this paragraph, we show that under some conditions on the kernels $K(t,s)$, $V(t,s)$ and the function $g(s,x)$, the functional integral equation (1.1) has a solution in $C([a,b])$. The following theorem ensures the existence of such a solution. Note that the proof of this theorem is based on the well-known Schaefer fixed point theorem that can be easily found in the literature, see for example [8, 20].

Theorem 2.1. *Consider the functional integral equation:*

$$x(t) = f(t) + \int_a^b K(t,s)x(s)\,ds + \int_a^b V(t,s)g(s,x(s))\,ds, \quad -\infty < a \le t \le b < +\infty, \tag{2.1}$$

where $f(\cdot) \in C([a,b])$. Assume that the function $g(s,x)$ satisfies the following conditions:

$$\sup\left(|g(s,x)|, \left|\frac{\partial g}{\partial x}(s,x)\right| \right) \le G(s)\phi(x), \tag{2.2}$$

for some measurable function $G(\cdot)$ and bounded function $\phi(\cdot)$. Assume that the kernels $K(t,s)$, $V(t,s)$ satisfy the following conditions:

$$|K(t,s)| \le K_1(t)K_2(s), \quad |V(t,s)| \le V_1(t)V_2(s), \tag{2.3}$$

for some continuous functions $K_1(\cdot)$, $V_1(\cdot)$, and $L^1([a,b])$ function $K_2(\cdot)$. Also, we assume *that the function $G(\cdot)V_2(\cdot) \in L^1([a,b])$. Finally, we assume that one of the following two conditions is satisfied:*

(c_1) $\|K_1\|_\infty \|K_2\|_1 < 1$,

(c_2) $K(t,s) = 0$, $\forall s > t$.

Under the above conditions, (2.1) has a solution in $C([a,b])$.

Proof. We first define the operator T by: $Tx(t) = f(t) + \int_a^b K(t,s)x(s)ds + \int_a^b V(t,s)g(s, x(s))ds$. By using (2.3) and by applying the dominated convergence theorem, one concludes that

$$\begin{aligned}
&\lim_{h\to 0} |Tx(t+h) - Tx(t)| \\
&\quad = \lim_{h\to 0} |f(t+h) - f(t)| + \int_a^b \lim_{h\to 0} |K(t+h,s) - K(t,s)| \|x\|_\infty ds \\
&\quad\quad + \int_a^b \lim_{h\to 0} |V(t+h,s) - V(t,s)| G(s)\|\phi\|_\infty ds = 0.
\end{aligned} \tag{2.4}$$

Hence, $Tx \in C([a,b])$ if $x \in C([a,b])$. Moreover, since

$$\begin{aligned}
&|Tx_n(t) - Tx(t)| \\
&\quad \le ||x_n - x||_\infty \left[\int_a^b |K(t,s)| ds + \int_a^b |V(t,s)| \left| \frac{\partial g}{\partial x}(s, \theta_s x_n + (1-\theta_s)x) \right| ds \right], \quad 0 < \theta_s < 1 \\
&\quad \le ||x_n - x||_\infty [||K_1||_\infty ||K_2||_1 + ||V_1||_\infty ||V_2 \cdot G||_1 \|\phi\|_\infty] = M||x_n - x||_\infty,
\end{aligned} \tag{2.5}$$

then $\lim_{n\to+\infty} \|Tx_n - Tx|_\infty = 0$, or equivalently, T is continuous over $C([a,b])$. Next, we prove that T is completely continuous on $E = C([a,b])$ or equivalently, it maps an arbitrary bounded set of the Banach space E into a compact set of E. By using Arzèla theorem [12], the complete continuity of T is ensured if $\{Tx_n;\ n \in N\}$ is equicontinuous and uniformly bounded for every uniformly bounded sequence $(x_n)_n$ of $C([a,b])$. This is done as follows:

$$\begin{aligned}
|Tx_n(t) - Tx_n(\tau)| \le |f(t) - f(\tau)| &+ \int_a^b |K(t,s) - K(\tau,s)| ||x_n(s)||_\infty ds \\
&+ \int_a^b |V(t,s) - V(\tau,s)| G(s)\|\phi\|_\infty ds.
\end{aligned} \tag{2.6}$$

Since $(x_n)_n$ is uniformly bounded, then by applying the dominated convergence theorem to the right-hand side of the previous inequality, one concludes that $\lim_{t\to\tau} |Tx_n(t) - Tx_n(\tau)| = 0$ independently of n or equivalently, $(Tx_n)_n$ is equicontinuous. Moreover, it is easy to see that $(Tx_n)_n$ is uniformly bounded whenever $(x_n)_n$ is a uniformly bounded sequence of $C([a,b])$. Hence, T is completely continuous. Finally, we prove the existence of a solution of (2.1). Since T is completly continuous, then by Schaefer fixed point theorem, we know that either:

(i) $x = \lambda Tx$ has a solution for $\lambda = 1$, or

(ii) the set $\mathscr{E} = \{u \in C([a,b]);\ \exists\lambda \in]0,1[,\ u = \lambda Tu\}$ is unbounded.

We prove that (ii) is not possible. Two cases are to be considered.

First case. We assume that (c_1) is satisfied and take $u \in \mathscr{E}$ satisfying $u = \lambda T u$ for some $0 < \lambda < 1$. Since $\int_a^b K_2(s)|u(s)|ds = |u(s^*)| \int_a^b K_2(s)ds$, for some $s^* \in [a,b]$, then it is easy to see that there exists a positive real number M such that

$$|u(t)| \leq M + |u(s^*)| \|K_1\|_\infty \|K_2\|_1, \quad \forall t \in [a,b]. \tag{2.7}$$

By using (c_1) and by taking $t = s^*$ in the previous inequality, one gets

$$|u(s^*)| \leq \frac{M}{1 - \|K_1\|_\infty \|K_2\|_1} = M'. \tag{2.8}$$

By substituting (2.8) in (2.7), one concludes that $|u(\cdot)|$ is bounded and consequently $\mathscr{E}$ is bounded.

Second case. We assume that condition (c_2) is satisfied. In this case, it is easy to see that for all $t \in [a,b]$, we have $|u(t)| \leq M + \|K_1\|_\infty \int_a^t K_2(s)|u(s)|ds$. By using Gronwall's inequality, one obtains $|u(t)| \leq M \exp(\|K_1\|_\infty \|K_2\|_1)$. Hence $u(\cdot)$ is bounded and consequently $\mathscr{E}$ is also bounded. □

An extension of the result of the previous theorem to a more general nonlinear integral equation is given by the following theorem. We skip the proof of this theorem because its techniques are similar to the techniques of the previous proof.

Theorem 2.2. *Consider the nonlinear integral equation:*

$$x(t) = f(t) + \int_a^b g(t,s,x(s))\,ds, \quad -\infty < a \leq t \leq b < +\infty, \tag{2.9}$$

where $f(\cdot) \in C([a,b])$. Assume that the function $g(t,s,x)$ satisfies the following conditions:

$$\begin{gathered} \sup\left(|g(t,s,x)|, \left|\frac{\partial g}{\partial t}(t,s,x)\right|\right) \leq V_1(t)V_2(s)\phi(|x|), \\ \left|\frac{\partial g}{\partial x}(t,s,x)\right| \leq V_1(t)V_2(s)\psi(|x|), \end{gathered} \tag{2.10}$$

where $V_1(\cdot) \in C([a,b])$, $V_2(\cdot) \in L^1([a,b])$, $\phi(\cdot)$ is positive and bounded over $[0,+\infty[$ and $\psi(\cdot)$ is positive and continuous over $[0,+\infty[$. Under the above conditions, (2.9) has a solution in $C([a,b])$.

Condition (2.10) with bounded $\phi(\cdot)$ is a limitation of the previous theorem. Nonetheless, by using a convenient new norm $\|\cdot\|_\mu$ and the Schauder fixed point theorem, one can prove the existence of continuous solutions of more general nonlinear integral equations with some weaker conditions. This is the subject of the next theorem.

THEOREM 2.3. *Consider the nonlinear integral equation*

$$x(t) = f(t) + \int_a^b g(t,s,x(s))\,ds, \quad -\infty < a \le t \le b < +\infty. \tag{2.11}$$

Assume that $f(\cdot)$ is bounded and $g(t,s,x)$ is continuous w.r.t. t and satisfies the following conditions:

$$|g(t,s,x)| \le V_1(t)V_2(s)\phi(|x|), \qquad \left|\frac{\partial g}{\partial x}(t,s,x)\right| \le V_1(t)V_2(s)\psi(|x|), \tag{2.12}$$

where $V_1(\cdot)$ is a measurable and bounded positive function, $\phi(\cdot)$ is a positive and measurable function satisfying the condition

$$\sup_{x\ge 0} \frac{\phi(x)}{x} = L < +\infty \tag{2.13}$$

and where $\psi(\cdot)$ is a positive and continuous function over $[0,+\infty[$. Moreover, assume that there exists a continuous, positive and bounded away from zero function $\mu(\cdot)$ satisying the following condition:

$$||V_1 \cdot \mu||_\infty \left|\left|\frac{V_2}{\mu}\right|\right|_1 < \frac{1}{L}. \tag{2.14}$$

Under the above conditions, the nonlinear integral equation (2.11) has a solution in $C([a,b])$.

Proof. We first mention that the function $\|\cdot\|_\mu$ defined on $X = C([a,b])$ by $\|x\|_\mu = \sup_{t\in[a,b]} |\mu(t)x(t)|$ is a norm on X. Next let $r \ge 0$ be a positive real number that will be fixed later on and define the subset B_r of X by $B_r = \{x \in C([a,b]);\ \|x\|_\mu \le r\}$. It is clear that B_r is a closed and convex subset of X. Let T be the operator defined on B_r by $Tx(t) = f(t) + \int_a^b g(t,s,x(s))ds$. It is easy to check that T maps bounded sets of B_r into relatively compact sets. By Schauder fixed point theorem see [20], to prove the existence of a solution of (2.11), it suffices to check that $T \in C(B_r, B_r)$. We first prove that $Tx(\cdot) \in C([a,b])$ whenever $x(\cdot) \in C([a,b])$. Let $(t_n)_n$ be a sequence in $[a,b]$ converging to t. Since $f(\cdot) \in C([a,b])$ and since for all $n \in N$, we have

$$|g(t_n,s,x(s))| \le V_1(t_n)V_2(s)M_{\phi,|x|} \le ||V_1||_\infty M_{\phi,|x|} V_2(s) \in L^1([a,b]), \tag{2.15}$$

where $M_{\phi,|x|}$ is a constant depending only on $\phi(\cdot)$ and $|x(\cdot)|$, then by applying the dominated convergence theorem, one concludes that

$$\lim_{n\to+\infty} Tx(t_n) = \lim_{n\to+\infty} f(t_n) + \int_a^b \lim_{n\to+\infty} g(t_n,s,x(s))\,ds = f(t) + \int_a^b g(t,s,x(s))\,ds. \tag{2.16}$$

Consequently, $Tx(\cdot) \in C([a,b])$. Next, we prove that T is continuous over B_r w.r.t. $\|\cdot\|_\mu$ norm. Let $(x_n)_n$ be a sequence of B_r converging to x in the $\|\cdot\|_\mu$ norm. Since $(B_r, \|\cdot\|_\mu)$

is complete, then $x \in B_r$. Moreover, we have

$$
\begin{aligned}
&\|Tx_n - Tx\|_\mu \\
&\quad = \sup_{t\in[a,b]} \left| \mu(t) \int_a^b \big(g(t,s,x_n(s)) - g(t,s,x(s)\big)\,ds \right| \\
&\quad \le \|\mu\|_\infty \int_a^b |x_n(s) - x(s)| \left| \frac{\partial g}{\partial x}\big(t,s,\theta_s x_n(s) + (1-\theta_s)x(s)\big)\,ds \right|, \quad \theta_s \in]0,1[\\
&\quad \le \|\mu\|_\infty \int_a^b V_1(t)\mu(s)|x_n(s) - x(s)| \frac{V_2(s)}{\mu(s)} \psi\big(|\theta_s x_n(s) + (1-\theta_s)x(s)|\big)\,ds, \quad \theta_s \in]0,1[\\
&\quad \le \|\mu\|_\infty \sup_{s\in[0,1]} \psi\big(|\theta_s x_n(s) + (1-\theta_s)x(s)|\big) \|x_n - x\|_\mu \left\| \frac{V_2}{\mu} \right\|_1 .
\end{aligned}
\tag{2.17}
$$

Since $\mu(\cdot)$ is continuous and bounded away from zero, then it is clear that convergence of $(x_n)_n$ to x in the $\|\cdot\|_\mu$ norm implies also the uniform convergence over $[a,b]$. Hence for all $n \in N$, $\forall s \in [0,1]$, one concludes that $|\theta_s x_n(s) + (1-\theta_s)x(s)|$ is contained in a compact set of $[0,+\infty[$. Moreover, since $\psi(\cdot)$ is continuous over $[0,+\infty[$, then one concludes that there exists a positive constant M_ψ such that $\psi(|\theta_s x_n(s) + (1-\theta_s)x(s)|) \le M_\psi$, $\forall s \in [0,1]$, $\forall n \in N$. Hence, the previous inequality becomes $\|Tx_n - Tx\|_\mu \le \|\mu\|_\infty M_\psi \|V_2/\mu\|_1 \|x_n - x\|_\mu$. Consequently, T is continuous over B_r. It remains to choose the positive real number r in such a way that $T(B_r) \subset B_r$. Let $x \in B_r$, then we have

$$
\begin{aligned}
\|Tx\|_\mu &\le \|\mu(t)f(t)\|_\infty + \left\| \mu(t)V_1(t) \int_a^b V_2(s)\phi(|x(s)|)\,ds \right\|_\infty \\
&\le \|f\|_\mu + \|V_1\|_\mu \left| \int_a^b \frac{V_2(s)}{\mu(s)} \mu(s)|x(s)| \frac{\phi(|x(s)|)}{|x(s)|}\,ds \right| \\
&\le \|f\|_\mu + \|V_1\|_\mu rL \left\| \frac{V_2}{\mu} \right\|_1 \le \|f\|_\mu + r\left(L\|V_1\|_\mu \left\| \frac{V_2}{\mu} \right\|_1 \right).
\end{aligned}
\tag{2.18}
$$

Hence, the condition $T(B_r) \subset B_r$ is satisfied for any positive real number r satisfying

$$
r \ge \frac{\|f\|_\mu}{1 - L\|V_1\|_\mu \|V_2/\mu\|_1} = r_0. \tag{2.19}
$$

By Schauder's fixed point theorem, one concludes that T has a fixed point in B_r for all $r \ge r_0$. □

Remark 2.4. In [18], a condition similar to the condition (2.14) has been used to prove the existence of a weakly continuous solution of a nonlinear integral equation. This solution is defined on $[0,1]$ and has values in a reflexive Banach space.

3. Existence and uniqueness results for a nonlinear integral equation

If in the Fredholm integral equation (2.11), we replace the integration bound b by the variable t, we obtain a nonlinear Volterra equation. We should mention that an extensive amount of work has been done in the existence and uniqueness of solutions of some special cases of Volterra integral equations, see for example [3, 6, 10, 18]. Under some conditions on the function $g(t,s,x)$ and by using the following Leray-Schauder principle, Theorem 3.2 ensures the existence of a solution of a nonlinear Volterra equation.

Theorem 3.1 (Leray-Schauder principle). *Let $(X, |\cdot|)$ be a Banach space and suppose that $T \in C(X,X)$ and compact. Suppose that any solution x of $x = \lambda Tx$, $0 \le \lambda \le 1$ satisfies the a priori bound $|x| \le M$ for some constant $M > 0$, then T has a fixed point.*

Theorem 3.2. *Consider the nonlinear Volterra integral equation*

$$x(t) = f(t) + \int_a^t g(t,s,x(s))\,ds, \quad -\infty < a \le t \le b < +\infty, \tag{3.1}$$

where f is continuous over $[a,b]$. Assume that $g(t,s,x)$ satisfies the following conditions:

$$|g(t,s,x)| \le V_1(t)V_2(s)\phi(|x|), \qquad \left|\frac{\partial g}{\partial x}(t,s,x)\right| \le V_1(t)V_2(s)\psi(|x|), \tag{3.2}$$

where $V_1(\cdot) \in C([a,b])$ and positive, $V_2(\cdot) \in L^1([a,b])$ and positive and where $\psi(\cdot)$ is a positive and continuous function over $[0,+\infty[$. Finally, we assume that the function $\phi(\cdot)$ is positive, continuous and satisfies the condition $\lim_{y\to+\infty}(\phi(y)/y) = L < +\infty$. Under the above conditions, (3.1) has a continuous solution over $[a,b]$.

Proof. Let $X = (C([a,b]), \|\cdot\|_\infty)$ denotes the Banach space of continuous functions over $[a,b]$ and define the operator T over X by $Tx(t) = f(t) + \int_a^t g(t,s,x(s))ds$. By using the conditions of the theorem, it is easy to check that $TX \subset X$ and T is compact. From Leray-Schauder principle, to prove the result of the theorem, it suffices to prove that T is continuous over X and any solution of $x = \lambda Tx$, $0 \le \lambda \le 1$ is bounded by the same constant $M > 0$. To prove the continuity of T over $C([a,b])$, it suffices to replace $\mu(t)$ by 1 in the proof of the continuity of the operator T of the previous theorem and follow the different steps of this proof. Next, we note that the condition $\lim_{y\to+\infty}(\phi(y)/y) = L < +\infty$ implies the existence of a positive real number $A > 0$ such that $|\phi(u)| \le (3/2)L = L'$, for all $u \ge A$. Let $x \in C([a,b])$ be a solution of $x = \lambda Tx$, for some $0 \le \lambda \le 1$, then we have

$$\begin{aligned}
|x(t)| &\le |\lambda|\,|f(t)| + |\lambda| \int_a^t |g(t,s,x(s))ds| \le \|f\|_\infty + \int_a^t V_1(t)V_2(s)\phi(|x(s)|)ds \\
&\le \|f\|_\infty + \|V_1\|_\infty \int_a^b V_2(s) \sup_{u\in[0,A]} \phi(u)ds + \|V_1\|_\infty \int_a^b V_2(s)L'|x(s)|ds \\
&\le \left[\|f\|_\infty + \sup_{u\in[0,A]} \phi(u)\|V_2\|_1\right] + \int_a^b (L'\|V_1\|_\infty V_2(s))|x(s)|ds \\
&\le M_1 + \int_a^b M_2 V_2(s)|x(s)|ds.
\end{aligned} \tag{3.3}$$

By using the general version of Gronwall's inequality together with the previous inequality, one concludes that $|x(t) \le M_1 \exp(M_2\|V_2\|_1) = M$. Since M_1 and M_2 do not depend on the solution x, then one concludes that the solutions of $x = \lambda Tx$, $0 \le \lambda \le 1$ are uniformly bounded by the same constant M. Finally, by using the Leray-Schauder principle, one concludes that T has a fixed point in $X = C([a,b])$ or equivalently, the nonlinear Volterra equation (3.1) has a continuous solution over $[a,b]$. □

The uniqueness of the solution of the nonlinear Volterra equation (3.1) is given by the following proposition.

Proposition 3.3. *Consider the nonlinear Volterra equation (3.1) and assume that $g(t,s,x)$ satisfies the conditions of Theorem 3.2 with $V_2(\cdot) \in (L^1 \cap L^p)([a,b])$ for some $p > 1$. Then (3.1) has a unique solution.*

Proof. The existence of a solution is ensured by Theorem 3.2. Next, note that in the proof of Theorem 3.2, we have shown that the continuous solutions of $x = Tx$ are uniformly bounded by the same constant M and consequently they are contained in a closed ball B_M given by

$$B_M = \{x \in C([a,b]);\ \|x\|_\infty \le M\}. \tag{3.4}$$

Hence, to prove the uniqueness of the solution of (3.1), it suffices to check that there exists $n_0 \in N$ such that T^{n_0} is a contraction in B_M. By using the notations of the proof of Theorem 3.2, one can easily check that for all $x, y \in C([a,b])$, we have

$$\begin{aligned} |Ty(t) - Tx(t)| &\le \|y-x\|_\infty \|V_1\|_\infty \|V_2\|_p (t-a)^{1/q} \sup_{u \in B_M} \psi(|u|) \\ &\le C\|y-x\|_\infty (t-a)^{1/q}. \end{aligned} \tag{3.5}$$

Similarly, one shows that

$$|T^2y(t) - T^2x(t)| \le C^2\|y-x\|_\infty \frac{1}{q+1}(t-a)^{1+1/q}. \tag{3.6}$$

Continuing in this manner, one can easily show that

$$|T^ny(t) - T^nx(t)| \le C^n\|y-x\|_\infty \prod_{i=1}^{n-1} \frac{1}{q+i}(t-a)^{n-1+1/q}. \tag{3.7}$$

Hence

$$\|T^ny - T^nx\|_\infty \le C^n\|y-x\|_\infty \left[\prod_{i=1}^{n-1} \frac{1}{q+i}\right](b-a)^{n-1+1/q}. \tag{3.8}$$

Since $\lim_{n\to+\infty}[\prod_{i=1}^{n-1}(1/(q+i))]C^n(b-a)^{n-1+1/q} = 0$, then there exists $n_0 \in N$ such that T^{n_0} is a contraction over B_M. Consequently, the fixed point of T^{n_0} is unique. Since a fixed point of T is also a fixed point of T^{n_0}, then one concludes that the fixed point of T is also unique and consequently, the solution of (3.1) is unique. □

4. Approximate solution of Volterra integral equation

In this last paragraph, we are interested in finding an approximate solution of Volterra integral equation of the type

$$x(t) = f(t) + \int_a^t g(t,s,x(s))\,ds, \quad -\infty < a \le t \le b < +\infty. \tag{4.1}$$

Note that the natural approach for finding an approximate solution of (4.1) is to use a quadrature scheme for the approximation of the integral term of (4.1), see [9, 14, 21]. In this section, we provide a new approach for approximating the solution of (4.1). It is described as follows. We first assume that (4.1) has a solution in $C^\alpha([a,b])$ for some $\alpha \ge 1$, $f \in C^1([a,b])$, the function $g(t,t,x)$ is continuous with respect to t and Lipschitzian w.r.t. x. Moreover, if $(t_n)_n$ is a sequence in $[a,b]$, then we assume that the functions $(\partial g/\partial t)(t_n,s,x)$ is equicontinuous w.r.t. s and Lipschitzian w.r.t. x. By using the above conditions and the standard existence proof for ordinary differential equation (O.D.E.) which is based on the successive approximations technique, one can easily check that the solution of (4.1) coincides with the unique solution of the following initial value problem obtained by differentiating (4.1):

$$x'(t) = f'(t) + \int_a^t \frac{\partial g}{\partial t}(t,s,x(s))\,ds + g(t,t,x(t)), \quad a \le t \le b,\ x(a) = f(a). \tag{4.2}$$

Hence the problem of finding an approximate solution of (4.1) is converted to the approximation of the solution of the integro-differential equation (4.2). Note that finding an approximate solution of the second problem is easier than for the first problem. This is due to the possibility of adapting existent approximation schemes from O.D.E. Our approximation scheme for solving (4.2) is described as follows. We first choose a uniform subdivision of $[a,b]$ denoted by $a = t_0 < t_1 < \cdots < t_N = b$ and let $h = t_{n+1} - t_n$, $0 \le n \le N-1$ be the stepsize of this subdivision. For $t_n \le t < t_n + 1$, we define a quadrature scheme $Q(t,x)$ for the approximation of the integral $\int_a^t (\partial g/\partial t)(t,s,x(s))ds$ as follows:

$$Q(t,x) = Q_1(t,x) + Q_2(t,x), \tag{4.3}$$

where $Q_1(t,x)$ is a qth order composite quadrature scheme for the approximation of $\int_a^{t_n} (\partial g \partial t)(t,s,x(s))ds$ constructed from a qth degree Lagrange interpolation polynomial obtained by the use of the grid points $t_i,\dots,t_{i-q+1}$ at the integration subinterval $[t_{i-1},t_i]$ for $1 \le i \le n$. Moreover, $Q_2(t,x)$ is a qth order quadrature scheme for the approximation of $\int_{t_n}^t (\partial g/\partial t)(t,s,x(s))ds$ constructed from a qth degree Lagrange extrapolation polynomial obtained by the use of the grid points $t_n,\dots,t_{n-q+1}$. Then, we consider a stable p-step method for solving the initial value problem $y'(t) = F(t,y(t))$, $y(a) = y_a$ given by

$$y_{n+1} = y_n + h\left[\sum_{i=0}^{p-1} \alpha_i F(t_{n-i}, y_{n-i})\right]. \tag{4.4}$$

If $\tilde{x}(t_{n+1})$ denotes the solution at $t = t_{n+1}$ of the following problem:

$$\tilde{x}(t) = f'(t) + Q(t,\tilde{x}) + g(t,t,\tilde{x}(t)), \quad a \le t \le b,\ \tilde{x}(a) = f'(a), \tag{4.5}$$

then an approximation $\widetilde{x}_{n+1}$ of $\widetilde{x}(t_{n+1})$ is given by:

$$\widetilde{x}_{n+1} = \widetilde{x}_n + h\left[\sum_{i=0}^{p-1} \alpha_i f'(t_{n-i}) + Q(t_{n-i},\widetilde{x}) + g(t_{n-i},t_{n-i},\widetilde{x})\right]. \tag{4.6}$$

In the sequel, we will denote by x_n, the approximation obtained via (4.6) of $x(t_n)$, where $x(t_n)$ denotes the exact value of the solution of (4.2) at $t = t_n$. The aim of the remaining of this paragraph is to find a global bound of the approximation error $|\widetilde{x}_n - x(t_n)|$, $n = 1,\ldots,N$. To this end, we first look for a bound of the local approximation error at the integration step $[t_n,t_{n+1}]$ and under the assumption that $\widetilde{x}_k = x(t_k)$ for all $k = 0,\ldots,n$. The order of this local error is given by the following proposition.

Proposition 4.1. *Assume that the function $g(t,s,x)$ is Lipschitzian w.r.t. x and the solution $\widetilde{x}(t)$ of (4.5) belongs to $C^{p+1}([t_n,t_{n+1}])$ for some positive integer p. Moreover, assume that the quadrature scheme $Q(t,x)$ satisfies the following condition:*

$$\sup_{t\in[a,b]} \left| \int_a^t \frac{\partial g}{\partial t}(t,s,x(s))\,ds - Q(t,x,h)\right| \le L_Q h^h. \tag{4.7}$$

Under the above conditions, we have $|x(t_{n+1}) - \widetilde{x}_{n+1}| = O(h^{\min(p,q)+1})$.

Proof. We first note that $|x(t_{n+1}) - \widetilde{x}_{n+1}| \le |x(t_{n+1}) - \widetilde{x}(t_{n+1})| + |\widetilde{x}(t_{n+1}) - \widetilde{x}_{n+1}|$. Since by hypothesis, $\widetilde{x}(\cdot) \in C^{p+1}([t_n,t_{n+1}])$ and $\widetilde{x}_{n+1}$ is an approximation of $\widetilde{x}(t_{n+1})$ obtained by the use of the p-step method (4.4), then we have

$$|\widetilde{x}(t_{n+1}) - \widetilde{x}_{n+1}| \le c\,|\widetilde{x}^{(p+1)}(\mu_{n+1})|\,h^{p+1} \le M_{n+1}h^{p+1}. \tag{4.8}$$

Here $\mu_{n+1} \in]t_n,t_{n+1}[$ and $M_{n+1} = c\sup_{t_n\le t\le t_{n+1}} |\widetilde{x}^{(p+1)}(\mu_{n+1})|$. It remains to bound the quantity $|x(t_{n+1}) - \widetilde{x}(t_{n+1})|$, this is done as follows. Since

$$\begin{aligned} x(t_{n+1}) - x(t_n) &= \int_{t_n}^{t_{n+1}} f'(t)\,dt + \int_{t_n}^{t_{n+1}} \left(\int_a^t \frac{\partial g}{\partial t}(t,s,x(s))\,ds\right)dt + \int_{t_n}^{t_{n+1}} g(t,t,x(t))\,dt \\ \widetilde{x}(t_{n+1}) - \widetilde{x}(t_n) &= \int_{t_n}^{t_{n+1}} f'(t)\,dt + \int_{t_n}^{t_{n+1}} Q(t,\widetilde{x},h)\,dt + \int_{t_n}^{t_{n+1}} g(t,t,x(t))\,dt, \end{aligned} \tag{4.9}$$

then

$$\begin{aligned} |x(t_{n+1}) - \widetilde{x}(t_{n+1})| \le{}& \int_{t_n}^{t_{n+1}} \left|\int_a^t \frac{\partial g}{\partial t}(t,s,x(s))\,ds - Q(t,\widetilde{x},h)\right| dt \\ &+ \int_{t_n}^{t_{n+1}} |g(t,t,x(t)) - g(t,t,\widetilde{x}(t))|\,dt. \end{aligned} \tag{4.10}$$

Since for all $t \in [t_n,t_{n+1}]$, $Q(t,\widetilde{x},h)$ depends on the values of $\widetilde{x}(\cdot)$ at the previous grid points $t_0 = a,t_1,\ldots,t_n$, and since by assumption $\widetilde{x}(t_i) = x(t_i)$, $i = 0,\ldots,n$, then

$$Q(t,\widetilde{x},h) = Q(t,x,h), \quad \forall t \in [t_n,t_{n+1}]. \tag{4.11}$$

Hence by using (4.7), one concludes that $\sup_{t\in[a,b]} |\int_a^t (\partial g/\partial t)(t,s,x(s))ds - Q(t,\tilde{x},h)| \leq L_Q h^q$ and consequently

$$\int_{t_n}^{t_{n+1}} \left| \int_a^t \frac{\partial g}{\partial t}(t,s,x(s))ds - Q(t,\tilde{x},h) \right| dt \leq L_Q h^{q+1}. \tag{4.12}$$

Moreover, since $g(t,s,x)$ is Lipschitzian w.r.t. x, then there exists a constant $L_g > 0$, such that

$$\int_{t_n}^{t_{n+1}} |g(t,t,x(t)) - g(t,t,\tilde{x}(t))| dt \leq L_g \int_{t_n}^{t_{n+1}} |x(t) - \tilde{x}(t)| dt. \tag{4.13}$$

By combining (4.12) and (4.13), one concludes that

$$|x(t_{n+1}) - \tilde{x}(t_{n+1})| \leq L_Q h^{q+1} + L_g \int_{t_n}^{t_{n+1}} |x(t) - \tilde{x}(t)| dt. \tag{4.14}$$

If $e(t) = |x(t) - \tilde{x}(t)|$, then the previous inequality is written as follows:

$$e(t_{n+1}) \leq L_Q h^{q+1} + \int_{t_n}^{t_{n+1}} L_g e(t) dt. \tag{4.15}$$

By applying Gronwall's inequality to (4.15), one obtains

$$e(t_{n+1}) \leq L_Q h^{q+1} \exp\left[\int_{t_n}^{t_{n+1}} L_g\, dt\right] = L_Q h^{q+1} e^{L_g h} = M' h^{q+1}. \tag{4.16}$$

Finally, by letting $M'_{n+1} = \max(M_{n+1}, M')$ and by combining (4.12) and (4.16), one obtains the following bound of the local approximation error $|x(t_{n+1}) - \tilde{x}_{n+1}| \leq M'_{n+1} h^{\min(p,q)+1}$. Hence the local approximation error of our proposed scheme is of order $O(h^{\min(p,q)+1})$. □

Finally, by removing the condition $x(t_i) = \tilde{x}(t_i)$ for $i = 0,\dots,n$, we obtain a global approximation error bound given by the following proposition.

Proposition 4.2. *Assume that $(\partial g/\partial t)(t,s,x)$ is Lipscitzian w.r.t. x and assume that the first p starting values x_i, $i = 0,\dots,p-1$ satisfy $\max_{i\leq p-1} |x(t_i) - x_i| = O(h^{\min(p,q)})$. Then under the hypotheses of the previous proposition, the global approximation error of our scheme is of order $O(h^{\min(p,q)})$.*

Proof. We first note that since $(\partial g/\partial t)(t,s,x)$ is Lipschitzian w.r.t. x, then the quatrature scheme $Q(t,x,h)$ for the approximation of $\int_a^t (\partial g/\partial t)(t,s,x(s))ds$ is also Lipschitzian w.r.t. x. Hence, there exists a constant $L'_Q > 0$ such $\sup_{t\in[a,b]} |Q(t,x,h) - Q(t,y,h)| \leq L'_Q \max_{i\leq n} |x_i - y_i|$. Next, let $F(t,x) = f'(t) + Q(t,x,h) + g(t,t,x)$ and note that

$$|F(t,x_n) - F(t,y_n)| \leq L'_Q \max_{i\leq n} |x_i - y_i| + L_g |x_n - y_n| \leq L_F \max_{i\leq n} |x_i - y_i|. \tag{4.17}$$

A bound of the global approximation error is given as follows:

$$\begin{aligned}
e_{n+1} &= \widetilde{x}_{n+1} - x(t_{n+1}) = \widetilde{x}_n + h\left[\sum_{i=0}^{p-1} \alpha_i F(t_{n-i}, \widetilde{x}_{n-i})\right] - x(t_{n+1}) \\
&= \widetilde{x}_n - x(t_n) + h\left[\sum_{i=0}^{p-1} \alpha_i \big(F(t_{n-i}, \widetilde{x}_{n-i}) - F(t_{n-i}, x(t_{n-i}))\big)\right] \\
&\quad + h\left[\sum_{i=0}^{p-1} \alpha_i F(t_{n-i}, x(t_{n-i}))\right] - [x(t_{n+1}) - x(t_n)]. \\
&= e_n + h\left[\sum_{i=0}^{p-1} \alpha_i \big(F(t_{n-i}, \widetilde{x}_{n-i}) - F(t_{n-i}, x(t_{n-i}))\big)\right] + E_n(h),
\end{aligned} \tag{4.18}$$

where $E_n(h) = -[x(t_{n+1}) - x(t_n) - h\sum_{i=0}^{p-1} \alpha_i F(t_{n-i}, x(t_{n-i}))] = x_{n+1} - x(t_{n+1})$. The previous proposition shows that $|E_n(h)| \le M'_{n+1} h^{\min(p,q)+1} \le M h^{\min(p,q)+1}$, where $M = \max\{M'_n;\ 1 \le n \le N\}$. Moreover, by using (4.17), one concludes that

$$\begin{aligned}
|e_{n+1}| &\le |e_n| + h\left[\sum_{i=0}^{p-1} \max |\alpha_i| L_F \max_{i\le n} |\widetilde{x}_{n-i} - x(t_{n-i})|\right] + M h^{\min(p,q)+1} \\
&\le |e_n| + p\cdot h\alpha \max_{i\le n} |\widetilde{x}_{n-i} - x(t_{n-i})| + M h^{\min(p,q)+1}, \\
\alpha = \max_{0\le i\le p-1} |\alpha_i| &\le (1 + h\alpha)\max_{i\le n} |e_i| + M h^{\min(p,q)+1}.
\end{aligned} \tag{4.19}$$

A simple induction on n shows that

$$\begin{aligned}
|e_{n+1}| &\le M h^{\min(p,q)+1} \frac{(1+h\alpha)^{n-p+1} - 1}{h\alpha} + (1+h\alpha)^{n-p+1} \max_{i\le p-1} |e_i| \\
&\le M h^{\min(p,q)+1} \frac{e^{\alpha(b-a)} - 1}{h\alpha} + e^{\alpha(b-a)} \max_{i\le p-1} |e_i| \le C h^{\min(p,q)}.
\end{aligned} \tag{4.20}$$

□

Remark 4.3. The proposed approximation scheme requires p starting values x_i, $i = 0,\dots, p-1$. A direct collocation method applied to (4.1) can be used to provide us with these appropriate starting values.

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