ON UNIVALENT SOLUTIONS OF THE BIHARMONIC EQUATION

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We analyze the univalence of the solutions of the biharmonic equation. In particular, we show that if F is a biharmonic map in the form $F(z) = r^2G(z)$, $|z| < 1$, where G is harmonic, then F is starlike whenever G is starlike. In addition, when $F(z) = r^2G(z) + K(z)$, $|z| < 1$, where G and K are harmonic, we show that F is locally univalent whenever G is starlike and K is orientation preserving.

1. Introduction

A continuous complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbf{C}$ is biharmonic if the laplacian of F is harmonic. Note that $\triangle F$ is harmonic in D, if F satisfies the biharmonic equation $\triangle(\triangle F) = 0$, where

$$\triangle = 4\frac{\partial^2}{\partial z \partial \overline{z}}. \tag{1.1}$$

In any simply connected domain D, it can be shown that F has the form

$$F = r^2G + K, \quad z = re^{i\theta}, \tag{1.2}$$

where G and K are harmonic in D. It is known [1, 2, 3] that H and G can be expressed as,

$$\begin{aligned} G &= g_1 + \overline{g_2}, \\ K &= k_1 + \overline{k_2}, \end{aligned} \tag{1.3}$$

where g_1, g_2, k_1, and k_2 are analytic in D (for details, see [1, 2, 3]). Lewy showed, see [2, 3], that a harmonic function W is locally univalent if Jacobian of W, J_W,

$$J_W = |W_z|^2 - |W_{\overline{z}}|^2 \neq 0. \tag{1.4}$$

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We say that a function W is orientation preserving if the

$$J_W = |W_z|^2 - |W_{\overline{z}}|^2 > 0. \tag{1.5}$$

Note that the composition $F \circ \phi$ of a harmonic function F with analytic function ϕ is harmonic, while this is not true when F is biharmonic. Without loss of generality, we consider the class of biharmonic mappings defined on the unit disc U.

Biharmonic functions arise in many physical situations, particularly in fluid dynamics and elasticity problems. Most important applications of the theory of functions of a complex variable were obtained in the plane theory of elasticity and in the approximate theory of plates subject to normal loading. That is, in cases when the solutions are biharmonic functions or functions associated with them. In linear elasticity, if the equations are formulated in terms of displacements for two-dimensional problems then the introduction of a stress function leads to a fourth-order equation of biharmonic type. For instance, the stress function is proved to be biharmonic for an elastically isotropic crystal undergoing phase transition, which follows spontaneous dilatation. Biharmonic functions also arise when dealing with transverse displacements of plates and shells. The biharmonic function can describe the deflection of a thin plate subjected to uniform loading over its surface with fixed edges. Biharmonic functions arise in fluid dynamics, particularly in Stokes flow problems (i.e., low-Reynolds-number flows). There is a wealth of applications for Stokes flow such as in engineering and biological transport phenomena (for details, see [5, 6, 7]). Fluid flow through a narrow pipe or channel, such as that used in micro-fluidics, involves low Reynolds number. Seepage flow through cracks and pulmonary alveolar blood flow can also be approximated by Stokes flow. Stokes flow also arises in flow through porous media, which have been long applied by civil engineers to groundwater movement. The industrial applications include the fabrication of microelectronic components, the effect of surface roughness on lubrication, the design of polymer dies and the development of peristaltic pumps for sensitive viscous materials. In natural systems, creeping flows are important in biomedical applications and studies of animal locomotion.

We consider in Section 2 the case when $F = r^2G$. We prove that when G maps ∂U univalently onto a bounded Jordan curve and that F maps U onto the inside of the Jordan curve then G is univalent inside U. In addition, we prove that F is starlike whenever G is starlike.

In Section 3, we give sufficient conditions that make $F = r^2G + K$ locally univalent. Some examples are given.

2. The case $F = r^2G$

First we establish the following.

Theorem 2.1. *Let D be a simply connected domain bounded by a Jordan curve. Let Ψ be a positively oriented one to one continuous function from the boundary of U, ∂U, onto the boundary of D, ∂D. Let G be the solution of the Dirichlet problem with respect to Ψ and F be the corresponding biharmonic functions, given by $F = r^2G$. If F is onto and $F(z) = 0$ only when $z = 0$ then G is one to one and onto.*

Proof. Let $a \in U$ and $b = F(a)$. Let J denote the line segment in D, containing b, intersecting ∂D exactly twice and given by: $\alpha u + \beta v = \gamma$, $\gamma \geq 0$. Let $I = F^{-1}(J)$. Note that $a \in I$ and I intersects ∂U exactly twice.

Next, we show that I is a simple curve in U. Let

$$\begin{aligned} G(z) &= u_1(z) + iv_1(z), \\ w(z) &= \alpha u_1(z) + \beta v_1(z). \end{aligned} \tag{2.1}$$

Then $w(z)$ is real harmonic in U, $w(z) = \gamma/|z|^2$ for all $z \in I$, $J_1 = G(I) = \{(u_1, v_1) : \alpha u_1(z) + \beta v_1(z) = \gamma/|z|^2\}$ and $G^{-1}(J_1) = I$.

First, note that I cannot contain an open disk, if it does then $w(z) = \gamma/|z|^2$ on that open disk. Simple calculation shows that the function $\gamma/|z|^2$ is subharmonic in $U\backslash\{0\}$. Hence w would be subharmonic and not harmonic on that open disk (assuming that the disk does not contain 0). This is impossible.

Second, we show that I cannot contain a loop without 0 in the inside, because if it does then, as w is harmonic, $\gamma/|z|^2$ is subharmonic and $w(z) = \gamma/|z|^2$ on that loop, $w(z) > \gamma/|z|^2$ inside the loop. Consequently, F maps the loop to J and the inside of that loop to the right of J. This is topologically impossible.

Third, note that if I loops around 0, then as the index of I around 0 is zero and the condition $F(z) = 0$ only when $z = 0$, I must contain a loop without 0 inside. Consequently by the second step, this is impossible.

Similarly, one can argue that I cannot contain a loop with no open inside. Hence I is a simple curve that divides U into two components. In one component $w(z) > \gamma/|z|^2$ and in the other one, $w(z) < \gamma/|z|^2$.

Rotate J counterclockwise. Since Ψ is positively oriented, the corresponding I will rotate counterclockwise.

Let

$$k(z) = w(z) + iV(z), \tag{2.2}$$

where V is the harmonic conjugate of w. Then $k(z)$ is analytic in U and, for $z \in I$,

$$\operatorname{Re} k(z) = w(z) = \gamma/|z|^2. \tag{2.3}$$

Note that in one side of I, $\operatorname{Re} k(z) > \gamma/|z|^2$ and in the other one, $\operatorname{Re} k(z) < \gamma/|z|^2$. Then, near a, k maps one side of I to one side of $k(I)$ and the other to the other side. Hence $k'(z) \neq 0$ at $z = a$ and

$$\alpha u_{1x}(a) + \beta v_{1x}(a) \neq 0 \tag{2.4}$$

or

$$\alpha u_{1y}(a) + \beta v_{1y}(a) \neq 0. \tag{2.5}$$

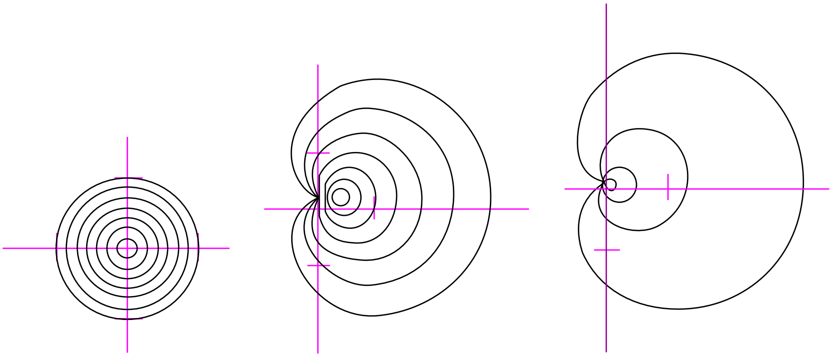

Figure 2.1

Choose $\alpha = v_{1y}(a)$ and $\beta = -u_{1y}(a)$. This makes (2.5) = 0, hence (2.4) $\neq 0$ and the determinant (the Jacobian of G) of the system is not zero. Since G is one to one on ∂U, the degree principle implies that G is one to one in U and, in addition, G is onto (see [2]). □

The following example shows that if G is one to one, it does not follow that F is one to one.

Example 2.2. Let $G(z) = ze^z + 0.4 + 0.2i$ and $F(z) = r^2G(z)$.

By drawing contours, using the $f(z)$ software (see Figure 2.1), we showed that in $|z| < 0.8$, G is one to one while F is not.

Following we show that when G is starlike then F is starlike and univalent.

Definition 2.3. We say that a biharmonic (harmonic) function is starlike on U if it is orientation-preserving, $F(0) = 0$, $F(z) \neq 0$ when $z \neq 0$ and the curve: $F(re^{\mathrm{it}})$ is starlike with respect to the origin for each $0 < r < 1$. In other words, $\partial \arg F(re^{\mathrm{it}})/\partial t = \operatorname{Re}(zF_z - \overline{z}F_{\overline{z}}/F) > 0$.

Remark 2.4. Note that starlike functions are univalent in U. The starlikeness condition implies that, for each $0 < r < 1$, F is univalent on the circle $|z| = r$ and as $F(0) = 0$, the orientation-preserving condition and the degree principle imply that it is univalent on $|z| \leq r$ and maps $|z| < r$ onto the inside of the curve $F(re^{\mathrm{it}})$.

If we denote the measure of starlikeness of F by

$$F_{\mathrm{st}} = \operatorname{Re}\frac{zF_z - \overline{z}F_{\overline{z}}}{F} \tag{2.6}$$

then we have the following lemma.

Lemma 2.5. *Let F be a biharmonic mapping in the unit disc U. If F is of the form $F = r^2G$, where G is a harmonic mapping in U then*

$$J_F = 2r^2|G|^2G_{\mathrm{st}} + r^4J_G. \tag{2.7}$$

Proof. Since $F = r^2 G$ this implies that

$$F_z = \overline{z}G + r^2 G_z,$$
$$F_{\overline{z}} = zG + r^2 G_{\overline{z}}. \tag{2.8}$$

Hence, it follows that

$$|F_z|^2 = (\overline{z}G + r^2 G_z)(z\overline{G} + r^2\overline{G_z}) = r^2|G|^2 + r^2\overline{z}\overline{G_z}G + r^2 zG_z\overline{G} + r^4|G_z|^2,$$
$$|F_{\overline{z}}|^2 = (zG + r^2 G_{\overline{z}})(\overline{z}\overline{G} + r^2\overline{G_{\overline{z}}}) = r^2|G|^2 + r^2 z\overline{G_{\overline{z}}}G + r^2\overline{z}G_{\overline{z}}\overline{G} + r^4|G_{\overline{z}}|^2. \tag{2.9}$$

Thus the Jacobian of F is given by

$$\begin{aligned}
J_F(z) &= |F_z|^2 - |F_{\overline{z}}|^2 \\
&= r^2 G(\overline{z}\overline{G_z} - z\overline{G_{\overline{z}}}) + r^2\overline{G}(zG_z - \overline{z}G_{\overline{z}}) + r^4\left(|G_z|^2 - |G_{\overline{z}}|^2\right) \\
&= r^2\left[\overline{z}G\overline{G_z} + z\overline{G}G_z - zG\overline{G_{\overline{z}}} - \overline{z}\overline{G}G_{\overline{z}}\right] + r^4\left(|G_z|^2 - |G_{\overline{z}}|^2\right) \\
&= r^2\left[2\operatorname{Re}(z\overline{G}G_z) - 2\operatorname{Re}(\overline{z}\overline{G}G_{\overline{z}})\right] + r^4\left(|G_z|^2 - |G_{\overline{z}}|^2\right) \\
&= 2r^2|G|^2\operatorname{Re}\left(\frac{zG_z - \overline{z}G_{\overline{z}}}{G}\right) + r^4\left(|G_z|^2 - |G_{\overline{z}}|^2\right) \\
&= 2r^2|G|^2 G_{\mathrm{st}} + r^4 J_G.
\end{aligned} \tag{2.10}$$

□

THEOREM 2.6. *Let F be a biharmonic mapping in the unit disc U. If F is of the form $F = r^2 G$, where G is a starlike harmonic mapping in U then F is starlike univalent in U.*

Proof. The lemma and the definition of starlikeness imply that $J_F(z) > 0$, for all $z \in U$. Hence F is orientation-preserving and locally univalent. Direct calculations imply that

$$F_{\mathrm{st}} = G_{\mathrm{st}}. \tag{2.11}$$

Hence F is starlike in U. □

COROLLARY 2.7. *$r^2 f(z)$ is starlike for all conformal starlike function f.*

Example 2.8. It is known that the harmonic Koebe function

$$k_0(z) = h(z) + \overline{g(z)}, \tag{2.12}$$

where

$$h(z) = \left(z - \frac{1}{2}z^2 + \frac{1}{6}z^3\right)(1-z)^{-3},$$
$$g(z) = \left(\frac{1}{2}z^2 + \frac{1}{6}z^3\right)(1-z)^{-3}, \tag{2.13}$$

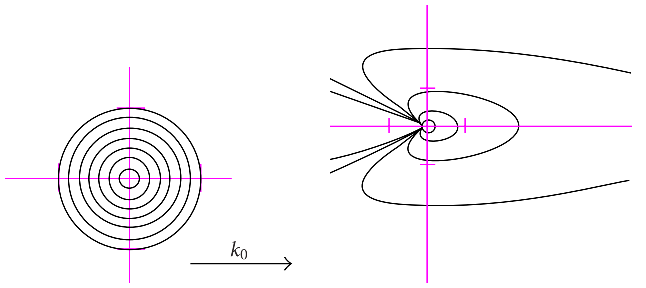

Figure 2.2

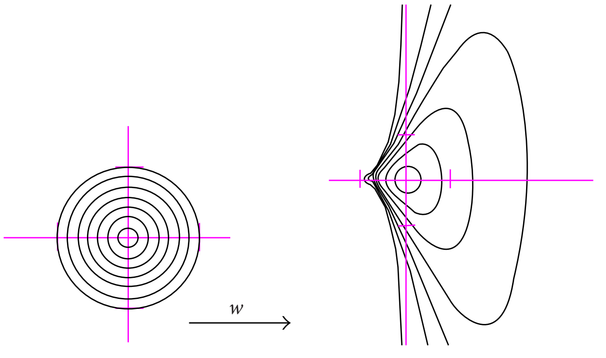

Figure 2.3

is starlike and hence the function r^2k_0 is also starlike. k_0 maps the unit disk univalently onto C minus the slit: $-\infty < t < -1/6$, as is shown in Figure 2.2.

Example 2.9. The following harmonic function:

$$w(z) = \operatorname{Re}(1+z)/(1-z) - 1 + i\operatorname{Im}\left(z/(1-z)^2\right), \tag{2.14}$$

is starlike and maps onto the half-plane $\{z : \operatorname{Re} z > -1\}$. Hence the function r^2w is also starlike. See Figure 2.3.

Remark 2.10. We do not know whether the converse of the theorem is true.

3. The general case

Theorem 3.1. *Let $F = r^2G + K$ be a biharmonic mapping in the unit disc U, where G is starlike harmonic and K is orientation preserving. If*

$$\operatorname{Re}\left[K_{\overline{z}}\overline{(r^2G)_{\overline{z}}}\right] < \operatorname{Re}\left[K_z\overline{(r^2G)_z}\right], \tag{3.1}$$

then $J_F(z) > 0$, for $z \neq 0$ and F is locally univalent.

Proof. Since $F = r^2G + K$, hence

$$F_{\bar{z}} = zG + r^2 G_{\bar{z}} + K_{\bar{z}}, \tag{3.2}$$

$$F_z = \bar{z}G + r^2 G_z + K_z. \tag{3.3}$$

Therefore, for $z \neq 0$,

$$\left| \frac{F_{\bar{z}}}{F_z} \right| = \left| \frac{G + \bar{z}G_{\bar{z}} + K_{\bar{z}}/z}{G + zG_z + K_z/\bar{z}} \right| = \left| \frac{1 + \bar{z}G_{\bar{z}}/G + K_{\bar{z}}/zG}{1 + zG_z/G + K_z/\bar{z}G} \right|. \tag{3.4}$$

Let

$$a = \frac{\bar{z}G_{\bar{z}}}{G}, \qquad b = \frac{K_{\bar{z}}}{zG}, \qquad c = \frac{zG_z}{G}, \qquad d = \frac{K_z}{\bar{z}G}. \tag{3.5}$$

Then (3.4) becomes

$$\left| \frac{F_{\bar{z}}}{F_z} \right| = \frac{|1 + a + b|^2}{|1 + c + d|^2}. \tag{3.6}$$

Note that,

$$\begin{aligned} |1 + a + b|^2 &= (1 + a + b)(1 + \bar{a} + \bar{b}) \\ &= 1 + 2\operatorname{Re}(a) + 2\operatorname{Re}(b) + 2\operatorname{Re}(a\bar{b}) + |a|^2 + |b|^2. \end{aligned} \tag{3.7}$$

Since G is starlike univalent harmonic mapping and K is an orientation preserving mapping, it follows that

$$\operatorname{Re} a = \operatorname{Re}\left(\frac{\bar{z}G_{\bar{z}}}{G}\right) < \operatorname{Re}\left(\frac{zG_z}{G}\right) = \operatorname{Re} c \tag{3.8}$$

and since $|G_{\bar{z}}| < |G_z|$ and $|K_{\bar{z}}| < |K_z|$,

$$|a| = \left| \frac{\bar{z}G_{\bar{z}}}{G} \right| < \left| \frac{zG_z}{G} \right| = |c|, \qquad |b| = \left| \frac{K_{\bar{z}}}{zG} \right| \leq \left| \frac{K_z}{\bar{z}G} \right| = |d|. \tag{3.9}$$

We have

$$\begin{aligned}
2\operatorname{Re} b + 2\operatorname{Re}(a\overline{b}) &= 2\operatorname{Re} b + 2\operatorname{Re}(\overline{a}b) = 2\operatorname{Re} b(1+\overline{a}),\\
b(1+\overline{a}) &= \frac{K_{\overline{z}}}{zG}\left(1+\frac{z\overline{G_{\overline{z}}}}{\overline{G}}\right),\\
d(1+\overline{c}) &= \frac{K_{z}}{\overline{z}G}\left(1+\frac{\overline{z}\overline{G_{z}}}{\overline{G}}\right).
\end{aligned}\tag{3.10}$$

Therefore,

$$\begin{aligned}
b(1+\overline{a}) - d(1+\overline{c}) &= \frac{K_{\overline{z}}}{zG}\left(1+\frac{z\overline{G_{\overline{z}}}}{\overline{G}}\right) - \frac{K_{z}}{\overline{z}G}\left(1+\frac{\overline{z}\overline{G_{z}}}{\overline{G}}\right)\\
&= \frac{K_{\overline{z}}}{zG} - \frac{K_{z}}{\overline{z}G} + \frac{K_{\overline{z}}\overline{G_{\overline{z}}}}{|G|^2} - \frac{K_{z}\overline{G_{z}}}{|G|^2}\\
&= \frac{\overline{z}\overline{G}K_{\overline{z}} - z\overline{G}K_{z}}{r^2|G|^2} + \frac{r^2K_{\overline{z}}\overline{G_{\overline{z}}} - r^2K_{z}\overline{G_{z}}}{r^2|G|^2}\\
&= \frac{1}{r^2|G|^2}\left[K_{\overline{z}}\left(\overline{z}\overline{G} + r^2\overline{G_{\overline{z}}}\right) - K_{z}\left(z\overline{G} + r^2\overline{G_{z}}\right)\right]\\
&= \frac{1}{r^2|G|^2}\left[K_{\overline{z}}\overline{(r^2G)_{\overline{z}}} - K_{z}\overline{(r^2G)_{z}}\right].
\end{aligned}\tag{3.11}$$

Hence, it follows from condition (3.1) that $\operatorname{Re}[b(1+\overline{a}) - d(1+\overline{c})] < 0$, consequently,

$$|1+a+b|^2 > |1+c+d|^2 \tag{3.12}$$

and the result follows. □

Remark 3.2. Consider the functions

$$\begin{aligned}
F_1(z) &= r^2z + z + \overline{z}/2 = r^2G_1(z) + K_1(z),\\
F_2(z) &= \frac{r^2z}{1-z} + z + \overline{z}/2 = r^2G_2(z) + K_2(z).
\end{aligned}\tag{3.13}$$

Direct calculations show that $F_1(z)$ and $F(z) = F_2(0.8z)$ satisfy all conditions of the theorem and in particular, condition (3.1).

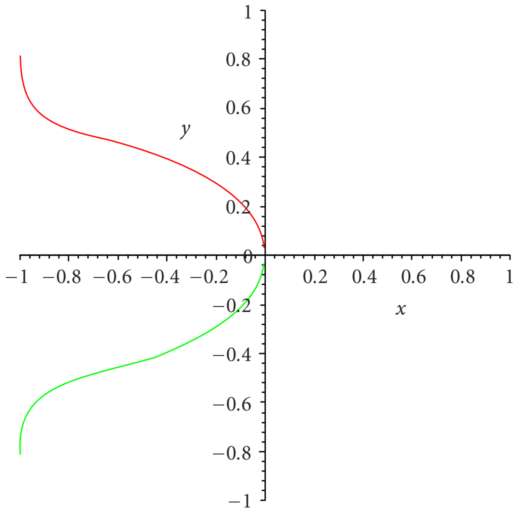

Figure 3.1

Remark 3.3. The theorem may not be true when G is not starlike. Here is an example: choose

$$\begin{aligned} G(z) &= 0.3\overline{z}^2 + z + 1, \\ F(z) &= r^2 G(z). \end{aligned} \tag{3.14}$$

G is orientation preserving and univalent in U. Suppose to the contrary that there are z_1 and z_2 in U, $z_1 \neq z_2$ and $G(z_1) = G(z_2)$ then it follows that $|z_1 + z_2| = 1/.3|(\overline{z_1 - z_2})/(z_1 - z_2)| > 3.3$, impossible. However, it was shown, using Maple, that the equation

$$J_F(z) = \left|1.3\overline{z}^3 + 2r^2\right|^2 - \left|0.9r^2\overline{z} + z^2 + z\right|^2 = 0 \tag{3.15}$$

has infinitely many zeros with graph as shown in Figure 3.1.

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