## Research Article

# Some Vector Inequalities for Continuous Functions of Self-Adjoint Operators in Hilbert Spaces 

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On utilizing the spectral representation of self-adjoint operators in Hilbert spaces, some inequalities for the composite operator $\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right]$, where $\operatorname{Sp}(A) \subseteq[m, M]$ and for various classes of continuous functions $f:[m, M] \rightarrow \mathbb{C}$ are given. Applications for the power function and the logarithmic function are also provided.

## 1. Introduction

Let $U$ be a self-adjoint operator on the complex Hilbert space $(H,\langle\cdot, \cdot\rangle)$ with the spectrum $\operatorname{Sp}(U)$ included in the interval $[m, M]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. Then, for any continuous function $f:[m, M] \rightarrow \mathbb{C}$, it is well known that we have the following spectral representation in terms of the Riemann-Stieltjes integral:

$$
\begin{equation*}
f(U)=\int_{m-0}^{M} f(\lambda) d E_{\lambda} \tag{1.1}
\end{equation*}
$$

which in terms of vectors can be written as

$$
\begin{equation*}
\langle f(U) x, y\rangle=\int_{m-0}^{M} f(\lambda) d\left\langle E_{\lambda} x, y\right\rangle \tag{1.2}
\end{equation*}
$$

for any $x, y \in H$. The function $g_{x, y}(\lambda):=\left\langle E_{\lambda} x, y\right\rangle$ is of bounded variation on the interval $[m, M]$ and

$$
\begin{equation*}
g_{x, y}(m-0)=0, \quad g_{x, y}(M)=\langle x, y\rangle, \tag{1.3}
\end{equation*}
$$

for any $x, y \in H$. It is also well known that $g_{x}(\lambda):=\left\langle E_{\lambda} x, x\right\rangle$ is monotonic nondecreasing and right continuous on $[m, M]$.

Utilising the spectral representation from (1.2), we have established the following Ostrowski-type vector inequality [1].

Theorem 1.1. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the $\operatorname{spectrum} \operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers $m<M$ and let $\left\{E_{\mathcal{\lambda}}\right\}_{\lambda}$ be its spectral family. If $f:[m, M] \rightarrow \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$, then one has the inequality

$$
\begin{align*}
&|f(s)\langle x, y\rangle-\langle f(A) x, y\rangle| \\
& \leq\left\langle E_{s} x, x\right\rangle^{1 / 2}\left\langle E_{s} y, y\right\rangle^{1 / 2} \bigvee_{m}^{s}(f) \\
&+\left\langle\left(1_{H}-E_{s}\right) x, x\right\rangle^{1 / 2}\left\langle\left(1_{H}-E_{s}\right) y, y\right\rangle^{1 / 2} \bigvee_{s}^{M}(f)  \tag{1.4}\\
& \leq\|x\|\|y\|\left(\frac{1}{2} \bigvee_{m}^{M}(f)+\frac{1}{2}\left|\bigvee_{m}^{s}(f)-\bigvee_{s}^{M}(f)\right|\right) \leq\|x\|\|y\| \bigvee_{m}^{M}(f),
\end{align*}
$$

for any $x, y \in H$ and for any $s \in[m, M]$.
Another result that compares the function of a self-adjoint operator with the integral mean is embodied in the following theorem [2].

Theorem 1.2. With the assumptions in Theorem 1.1 one has the inequalities

$$
\begin{align*}
& \left|\langle x, y\rangle \cdot \frac{1}{M-m} \int_{m}^{M} f(s) d s-\langle f(A) x, y\rangle\right| \\
& \leq \bigvee_{m}^{M}(f) \max _{t \in[m, M]}\left[\frac{M-t}{M-m}\left\langle E_{t} x, x\right\rangle^{1 / 2}\left\langle E_{t} y, y\right\rangle^{1 / 2}+\frac{t-m}{M-m}\left\langle\left(1_{H}-E_{t}\right) x, x\right\rangle^{1 / 2}\left\langle\left(1_{H}-E_{t}\right) y, y\right\rangle^{1 / 2}\right] \\
& \leq\|x\|\|y\| \bigvee_{m}^{M}(f), \tag{1.5}
\end{align*}
$$

for any $x, y \in H$.
The trapezoid version of the above result has been obtained in [3] and is as follows.

Theorem 1.3. With the assumptions in Theorem 1.1 one has the inequalities

$$
\begin{align*}
& \left|\frac{f(M)+f(m)}{2} \cdot\langle x, y\rangle-\langle f(A) x, y\rangle\right| \\
& \quad \leq \frac{1}{2} \max _{\lambda \in[m, M]}\left[\left\langle E_{\lambda} x, x\right\rangle^{1 / 2}\left\langle E_{\curlywedge} y, y\right\rangle^{1 / 2}+\left\langle\left(1_{H}-E_{\curlywedge}\right) x, x\right\rangle^{1 / 2}\left\langle\left(1_{H}-E_{\curlywedge}\right) y, y\right\rangle^{1 / 2}\right] \bigvee_{m}^{M}(f) \\
& \quad \leq \frac{1}{2}\|x\|\|y\| \bigvee_{m}^{M}(f), \tag{1.6}
\end{align*}
$$

for any $x, y \in H$.
For various inequalities for functions of self-adjoint operators, see [4-8]. For recent results see [1, 9-12].

In this paper, we investigate the quantity

$$
\begin{equation*}
\left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right|, \tag{1.7}
\end{equation*}
$$

where $x, y$ are vectors in the Hilbert space $H$ and $A$ is a self-adjoint operator with $\operatorname{Sp}(A) \subseteq$ $[m, M]$, and provide different bounds for some classes of continuous functions $f:[m, M] \rightarrow$ $\mathbb{C}$. Applications for some particular cases including the power and logarithmic functions are provided as well.

## 2. Some Vector Inequalities

The following representation in terms of the spectral family is of interest in itself.
Lemma 2.1. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the spectrum $\operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers $m<M$ and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:[m, M] \rightarrow \mathbb{C}$ is a continuous function on $[m, M]$ with $f(M) \neq f(m)$, then one has the representation

$$
\begin{align*}
& \frac{1}{[f(M)-f(m)]^{2}}\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] \\
& \quad=\frac{1}{f(M)-f(m)} \int_{m-0}^{M}\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right)\left(E_{t}-\frac{1}{2} 1_{H}\right) d f(t) . \tag{2.1}
\end{align*}
$$

Proof. We observe

$$
\begin{align*}
& \frac{1}{f(M)-f(m)} \int_{m-0}^{M}\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right)\left(E_{t}-\frac{1}{2} 1_{H}\right) d f(t) \\
&= \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t}^{2} d f(t) \\
&-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s) \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)  \tag{2.2}\\
&-\frac{1}{2} \int_{m-0}^{M} E_{t} d f(t)+\frac{1}{2} \int_{m-0}^{M} E_{s} d f(s) \\
&= \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t}^{2} d f(t)-\left[\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)\right]^{2}
\end{align*}
$$

which is an equality of interest in itself.
Since $E_{t}$ are projections, we have $E_{t}^{2}=E_{t}$ for any $t \in[m, M]$ and then we can write

$$
\begin{align*}
& \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t}^{2} d f(t)-\left[\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)\right]^{2} \\
& \quad=\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)-\left[\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)\right]^{2}  \tag{2.3}\\
& =\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)\left[1_{H}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)\right]
\end{align*}
$$

Integrating by parts in the Riemann-Stieltjes integral and utilizing the spectral representation (1.1), we have

$$
\begin{gather*}
\int_{m-0}^{M} E_{t} d f(t)=f(M) 1_{H}-f(A) \\
1_{H}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{t} d f(t)=\frac{f(A)-f(m) 1_{H}}{f(M)-f(m)} \tag{2.4}
\end{gather*}
$$

which together with (2.3) and (2.2) produce the desired result (2.1).
The following vector version may be stated as well.

Corollary 2.2. With the assumptions of Lemma 2.1 one has the equality

$$
\begin{align*}
& \left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle \\
& \quad=[f(M)-f(m)] \int_{m-0}^{M}\left\langle\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right) x,\left(E_{t}-\frac{1}{2} 1_{H}\right) y\right\rangle d f(t), \tag{2.5}
\end{align*}
$$

for any $x, y \in[m, M]$.
The following result that provides some bounds for continuous functions of bounded variation may be stated as well.

Theorem 2.3. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the spectrum $\operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers $m<M$, and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:[m, M] \rightarrow \mathbb{C}$ is a continuous function of bounded variation on $[m, M]$ with $f(M) \neq f(m)$, then we have the inequality

$$
\begin{align*}
& \left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq \frac{1}{2}\|y\||f(M)-f(m)| \bigvee_{m}^{M}(f) \\
& \quad \times \sup _{t \in[m, M]}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right\|  \tag{2.6}\\
& \quad \leq \frac{1}{2}\|x\|\|y\|\left[\bigvee_{m}^{M}(f)\right]^{2},
\end{align*}
$$

for any $x, y \in H$.
Proof. It is well known that if $p:[a, b] \rightarrow \mathbb{C}$ is a bounded function, $v:[a, b] \rightarrow \mathbb{C}$ is of bounded variation, and the Riemann-Stieltjes integral $\int_{a}^{b} p(t) d v(t)$ exists, then the following inequality holds:

$$
\begin{equation*}
\left|\int_{a}^{b} p(t) d v(t)\right| \leq \sup _{t \in[a, b]}|p(t)| \bigvee_{a}^{b}(v) \tag{2.7}
\end{equation*}
$$

where $\bigvee_{a}^{b}(v)$ denotes the total variation of $v$ on $[a, b]$.

Utilising this property and the representation (2.5), we have by the Schwarz inequality in Hilbert space $H$ that

$$
\begin{align*}
& \left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq|f(M)-f(m)| \bigvee_{m}^{M}(f) \\
& \quad \times \sup _{t \in[m, M]}\left|\left\langle\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right) x,\left(E_{t}-\frac{1}{2} 1_{H}\right) y\right\rangle\right|  \tag{2.8}\\
& \leq|f(M)-f(m)| \bigvee_{m}^{M}(f) \\
& \quad \times \sup _{t \in[m, M]}\left[\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|\left\|E_{t} y-\frac{1}{2} y\right\|\right]
\end{align*}
$$

for any $x, y \in[m, M]$.
Since $E_{t}$ are projections, in this case we have

$$
\begin{align*}
\left\|E_{t} y-\frac{1}{2} y\right\|^{2} & =\left\|E_{t} y\right\|^{2}-\left\langle E_{t} y, y\right\rangle+\frac{1}{4}\|y\|^{2}  \tag{2.9}\\
& =\left\langle E_{t}^{2} y, y\right\rangle-\left\langle E_{t} y, y\right\rangle+\frac{1}{4}\|y\|^{2}=\frac{1}{4}\|y\|^{2}
\end{align*}
$$

then from (2.8), we deduce the first part of (2.6).
Now, by the same property (2.7) for vector-valued functions $p$ with values in Hilbert spaces, we also have

$$
\begin{align*}
& \left\|[f(M)-f(m)] E_{t} x-\int_{m-0}^{M} E_{s} x d f(s)\right\| \\
& \quad=\left\|\int_{m-0}^{M}\left(E_{t} x-E_{s} x\right) d f(s)\right\| \leq \bigvee_{m}^{M}(f) \sup _{s \in[m, M]}\left\|E_{t} x-E_{s} x\right\|, \tag{2.10}
\end{align*}
$$

for any $t \in[m, M]$ and $x \in H$.
Since $0 \leq E_{t} \leq 1_{H}$ in the operator order, then $-1_{H} \leq E_{t}-E_{s} \leq 1$ which gives that $-\|x\|^{2} \leq\left\langle\left(E_{t}-E_{s}\right) x, x\right\rangle \leq\|x\|^{2}$, that is, $\left|\left\langle\left(E_{t}-E_{s}\right) x, x\right\rangle\right| \leq\|x\|^{2}$ for any $x \in H$, which implies that $\left\|E_{t}-E_{s}\right\| \leq 1$ for any $t, s \in[m, M]$. Therefore, $\sup _{s \in[m, M]}\left\|E_{t} x-E_{s} x\right\| \leq\|x\|$ which together with (2.10) prove the last part of (2.6).

The case of Lipschitzian functions is as follows.

Theorem 2.4. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the $\operatorname{spectrum} \operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers $m<M$, and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:[m, M] \rightarrow \mathbb{C}$ is a Lipschitzian function with the constant $L>0$ on $[m, M]$ and with $f(M) \neq f(m)$, then one has the inequality

$$
\begin{align*}
& \left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq \frac{1}{2} L\|y\||f(M)-f(m)| \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d t \\
& \quad \leq \frac{1}{2} L^{2}\|y\| \iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\| d s d t  \tag{2.11}\\
& \quad \leq \frac{\sqrt{2}}{2} L^{2}\|y\|(M-m)\langle A x-m x, M x-A x\rangle^{1 / 2} \\
& \quad \leq \frac{\sqrt{2}}{4} L^{2}\|y\|\|x\|(M-m)^{2},
\end{align*}
$$

for any $x, y \in H$.
Proof. Recall that if $p:[a, b] \rightarrow \mathbb{C}$ is a Riemann integrable function and $v:[a, b] \rightarrow \mathbb{C}$ is Lipschitzian with the constant $L>0$, that is,

$$
\begin{equation*}
|f(s)-f(t)| \leq L|s-t| \quad \text { for any } t, s \in[a, b], \tag{2.12}
\end{equation*}
$$

then the Riemann-Stieltjes integral $\int_{a}^{b} p(t) d v(t)$ exists and the following inequality holds:

$$
\begin{equation*}
\left|\int_{a}^{b} p(t) d v(t)\right| \leq L \int_{a}^{b}|p(t)| d t . \tag{2.13}
\end{equation*}
$$

Now, on applying this property of the Riemann-Stieltjes integral, then we have from the representation (2.5) that

$$
\begin{align*}
\left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
\quad \leq|f(M)-f(m)| \int_{m-0}^{M}\left|\left\langle\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right) x,\left(E_{t}-\frac{1}{2} 1_{H}\right) y\right\rangle\right| d f(t), \\
\quad \leq L|f(M)-f(m)| \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|\left\|E_{t} y-\frac{1}{2} y\right\| d t \\
\quad=\frac{1}{2} L\|y\||f(M)-f(m)| \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d t \tag{2.14}
\end{align*}
$$

for any $x, y \in H$ and the first inequality in (2.11) is proved.

Further, observe that

$$
\begin{align*}
& |f(M)-f(m)| \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d t \\
& \quad=\int_{m-0}^{M}\left\|[f(M)-f(m)] E_{t} x-\int_{m-0}^{M} E_{s} x d f(s)\right\| d t  \tag{2.15}\\
& \quad=\int_{m-0}^{M}\left\|\int_{m-0}^{M}\left(E_{t} x-E_{s} x\right) d f(s)\right\| d t
\end{align*}
$$

for any $x \in H$.
If we use the vector-valued version of the property (2.13), then we have

$$
\begin{equation*}
\int_{m-0}^{M}\left\|\int_{m-0}^{M}\left(E_{t} x-E_{s} x\right) d f(s)\right\| d t \leq L \iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\| d s d t \tag{2.16}
\end{equation*}
$$

for any $x \in H$ and the second part of (2.11) is proved.
Further on, by applying the double-integral version of the Cauchy-BuniakowskiSchwarz inequality, we have

$$
\begin{equation*}
\iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\| d s d t \leq(M-m)\left(\iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\|^{2} d s d t\right)^{1 / 2} \tag{2.17}
\end{equation*}
$$

for any $x \in H$.
Now, by utilizing the fact that $E_{s}$ are projections for each $s \in[m, M]$, then we have

$$
\begin{align*}
& \iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\|^{2} d s d t \\
& \quad=2\left[(M-m) \int_{m-0}^{M}\left\|E_{t} x\right\|^{2} d t-\left\|\int_{m-0}^{M} E_{t} x d t\right\|^{2}\right]  \tag{2.18}\\
& \quad=2\left[(M-m) \int_{m-0}^{M}\left\langle E_{t} x, x\right\rangle d t-\left\|\int_{m-0}^{M} E_{t} x d t\right\|^{2}\right]
\end{align*}
$$

for any $x \in H$.

If we integrate by parts and use the spectral representation (1.2), then we get

$$
\begin{equation*}
\int_{m-0}^{M}\left\langle E_{t} x, x\right\rangle d t=\langle M x-A x, x\rangle, \quad \int_{m-0}^{M} E_{t} x d t=M x-A x \tag{2.19}
\end{equation*}
$$

and by (2.18), we then obtain the following equality of interest:

$$
\begin{equation*}
\iint_{m-0}^{M}\left\|E_{t} x-E_{s} x\right\|^{2} d s d t=2\langle A x-m x, M x-A x\rangle, \tag{2.20}
\end{equation*}
$$

for any $x \in H$.
On making use of (2.20) and (2.17), we then deduce the third part of (2.11).
Finally, by utilizing the elementary inequality in inner product spaces

$$
\begin{equation*}
\operatorname{Re}\langle a, b\rangle \leq \frac{1}{4}\|a+b\|^{2}, \quad a, b \in H, \tag{2.21}
\end{equation*}
$$

we also have that

$$
\begin{equation*}
\langle A x-m x, M x-A x\rangle \leq \frac{1}{4}(M-m)^{2}\|x\|^{2}, \tag{2.22}
\end{equation*}
$$

for any $x \in H$, which proves the last inequality in (2.11).
The case of nondecreasing monotonic functions is as follows.
Theorem 2.5. Let A be a self-adjoint operator in the Hilbert space $H$ with the spectrum $\operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers $m<M$, and let $\left\{E_{\lambda}\right\}_{\lambda}$ be its spectral family. If $f:[m, M] \rightarrow \mathbb{R}$ is a monotonic nondecreasing function on $[m, M]$, then one has the inequality

$$
\begin{align*}
& \left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq \frac{1}{2}\|y\|[f(M)-f(m)] \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d f(t) \\
& \quad \leq \frac{1}{2}\|y\|[f(M)-f(m)]\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, x\right\rangle^{1 / 2}  \tag{2.23}\\
& \quad \leq \frac{1}{4}\|y\|\|x\|[f(M)-f(m)]^{2},
\end{align*}
$$

for any $x, y \in H$.

Proof. From the theory of Riemann-Stieltjes integral, it is also well known that if $p:[a, b] \rightarrow$ $\mathbb{C}$ is of bounded variation and $v:[a, b] \rightarrow \mathbb{R}$ is continuous and monotonic nondecreasing, then the Riemann-Stieltjes integrals $\int_{a}^{b} p(t) d v(t)$ and $\int_{a}^{b}|p(t)| d v(t)$ exist and

$$
\begin{equation*}
\left|\int_{a}^{b} p(t) d v(t)\right| \leq \int_{a}^{b}|p(t)| d v(t) \tag{2.24}
\end{equation*}
$$

Now, on applying this property of the Riemann-Stieltjes integral, we have from the representation (2.5) that

$$
\begin{align*}
& \left|\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq[f(M)-f(m)] \int_{m-0}^{M}\left|\left\langle\left(E_{t}-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} d f(s)\right) x,\left(E_{t}-\frac{1}{2} 1_{H}\right) y\right\rangle\right| d f(t), \\
& \quad \leq[f(M)-f(m)] \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|\left\|E_{t} y-\frac{1}{2} y\right\| d f(t) \\
& \quad=\frac{1}{2}\|y\|[f(M)-f(m)] \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d f(t), \tag{2.25}
\end{align*}
$$

for any $x, y \in H$, which proves the first inequality in (2.23).
On utilizing the Cauchy-Buniakowski-Schwarz-type inequality for the RiemannStieltjes integral of monotonic nondecreasing integrators, we have

$$
\begin{align*}
& \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\| d f(t) \\
& \quad \leq\left[\int_{m-0}^{M} d f(t)\right]^{1 / 2}\left[\int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|^{2} d f(t)\right]^{1 / 2}, \tag{2.26}
\end{align*}
$$

for any $x, y \in H$.

Observe that

$$
\begin{align*}
\int_{m-0}^{M} \| & E_{t} x- \\
=\int_{m-0}^{M} & {\left[\left\|E_{t} x\right\|^{2}-2 \operatorname{Re}\left\langle E_{t} x, \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s) \|^{2} d f(t)\right.\right.} \\
& \left.\left.\quad+\| \frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\rangle \|^{2}\right] d f(t) \\
= & {[f(M)-f(m)]\left[\frac{1}{f(M)-f(m)} \int_{m-0}^{M}\left\|E_{t} x\right\|^{2} d f(t)-\left\|\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|^{2}\right] } \tag{2.27}
\end{align*}
$$

and, integrating by parts in the Riemann-Stieltjes integral, we have

$$
\begin{align*}
\int_{m-0}^{M}\left\|E_{t} x\right\|^{2} d f(t) & =\int_{m-0}^{M}\left\langle E_{t} x, E_{t} x\right\rangle d f(t)=\int_{m-0}^{M}\left\langle E_{t} x, x\right\rangle d f(t) \\
& =f(M)\|x\|^{2}-\int_{m-0}^{M} f(t) d\left\langle E_{t} x, x\right\rangle  \tag{2.28}\\
& =f(M)\|x\|^{2}-\langle f(A) x, x\rangle=\left\langle\left[f(M) 1_{H}-f(A)\right] x, x\right\rangle, \\
\int_{m-0}^{M} E_{s} x d f(s) & =f(M) x-f(A) x,
\end{align*}
$$

for any $x \in H$.
On making use of the equalities (2.28), we have

$$
\begin{align*}
& \frac{1}{f(M)-f(m)} \int_{m-0}^{M}\left\|E_{t} x\right\|^{2} d f(t)-\left\|\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|^{2} \\
&=\frac{1}{[f(M)-f(m)]^{2}}\left[[f(M)-f(m)]\left\langle\left[f(M) 1_{H}-f(A)\right] x, x\right\rangle-\|f(M) x-f(A) x\|^{2}\right] \\
&=\frac{[f(M)-f(m)]\left\langle\left[f(M) 1_{H}-f(A)\right] x, x\right\rangle-\langle f(M) x-f(A) x, f(M) x-f(A) x\rangle}{[f(M)-f(m)]^{2}} \\
&=\frac{[f(M)-f(m)]\left\langle\left[f(M) 1_{H}-f(A)\right] x, x\right\rangle-\langle f(M) x-f(A) x, f(M) x-f(A) x\rangle}{[f(M)-f(m)]^{2}} \\
&=\frac{\langle f(M) x-f(A) x, f(A) x-f(m) x\rangle}{[f(M)-f(m)]^{2}}, \tag{2.29}
\end{align*}
$$

for any $x \in H$.

Therefore, we obtain the following equality of interest in itself as well:

$$
\begin{align*}
& \frac{1}{f(M)-f(m)} \int_{m-0}^{M}\left\|E_{t} x-\frac{1}{f(M)-f(m)} \int_{m-0}^{M} E_{s} x d f(s)\right\|^{2} d f(t) \\
& =\frac{\langle f(M) x-f(A) x, f(A) x-f(m) x\rangle}{[f(M)-f(m)]^{2}}  \tag{2.30}\\
& =\frac{\left\langle\left[f(M) 1_{H}-f(A)\right]\left[f(A)-f(m) 1_{H}\right] x, x\right\rangle}{[f(M)-f(m)]^{2}}
\end{align*}
$$

for any $x \in H$
On making use of the inequality (2.26), we deduce the second inequality in (2.23).
The last part follows by (2.21), and the details are omitted.

## 3. Applications

We consider the power function $f(t):=t^{p}$, where $p \in \mathbb{R} \backslash\{0\}$ and $t>0$. The following power inequalities hold.

Proposition 3.1. Let $A$ be a self-adjoint operator in the Hilbert space $H$ with the spectrum $\operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers with $0 \leq m<M$.

If $p>0$, then for any $x, y \in H$,

$$
\begin{align*}
& \left|\left\langle\left(M^{p} 1_{H}-A^{p}\right)\left(A^{p}-m^{p} 1_{H}\right) x, y\right\rangle\right| \\
& \quad \leq \frac{\sqrt{2}}{2} B_{p}^{2}\|y\|(M-m)\langle A x-m x, M x-A x\rangle^{1 / 2}  \tag{3.1}\\
& \quad \leq \frac{\sqrt{2}}{4} B_{p}^{2}\|y\|\|x\|(M-m)^{2}
\end{align*}
$$

where

$$
\begin{align*}
& \quad B_{p}=p \times \begin{cases}M^{p-1}, & \text { if } p \geq 1, \\
m^{p-1}, & \text { if } 0<p<1, m>0,\end{cases} \\
& \left|\left\langle\left(A^{-p}-M^{-p} 1_{H}\right)\left(m^{-p} 1_{H}-A^{-p}\right) x, y\right\rangle\right|  \tag{3.2}\\
& \leq \frac{\sqrt{2}}{2} C_{p}^{2}\|y\|(M-m)\langle A x-m x, M x-A x\rangle^{1 / 2} \\
& \leq \frac{\sqrt{2}}{4} C_{p}^{2}\|y\|\|x\|(M-m)^{2},
\end{align*}
$$

where

$$
\begin{equation*}
C_{p}=p m^{-p-1}, \quad m>0 . \tag{3.3}
\end{equation*}
$$

The proof follows from Theorem 2.4 applied for the power function.
Proposition 3.2. Let A be a self-adjoint operator in the Hilbert space $H$ with the $\operatorname{spectrum} \operatorname{Sp}(A) \subseteq$ [ $m, M$ ] for some real numbers with $0 \leq m<M$.

If $p>0$, then for any $x, y \in H$

$$
\begin{align*}
& \left|\left\langle\left(M^{p} 1_{H}-A^{p}\right)\left(A^{p}-m^{p} 1_{H}\right) x, y\right\rangle\right| \\
& \quad \leq \frac{1}{2}\|y\|\left(M^{p}-m^{p}\right)\left\langle\left(M^{p} 1_{H}-A^{p}\right)\left(A^{p}-m^{p} 1_{H}\right) x, x\right\rangle^{1 / 2} \\
& \quad \leq \frac{1}{4}\|y\|\|x\|\left(M^{p}-m^{p}\right)^{2}, \\
& \left|\left\langle\left(A^{-p}-M^{-p} 1_{H}\right)\left(m^{-p} 1_{H}-A^{-p}\right) x, y\right\rangle\right|  \tag{3.4}\\
& \quad \leq \frac{1}{2}\|y\|\left(m^{-p}-M^{-p}\right)\left\langle\left(A^{-p}-M^{-p} 1_{H}\right)\left(m^{-p} 1_{H}-A^{-p}\right) x, x\right\rangle^{1 / 2} \\
& \quad \leq \frac{1}{4}\|y\|\|x\|\left(m^{-p}-M^{-p}\right)^{2} .
\end{align*}
$$

The proof follows from Theorem 2.5.
Now, consider the logarithmic function $f(t)=\ln t, t>0$. We have the following
Proposition 3.3. Let A be a self-adjoint operator in the Hilbert space $H$ with the spectrum $\operatorname{Sp}(A) \subseteq$ $[m, M]$ for some real numbers with $0<m<M$. Then one has the inequalities

$$
\begin{align*}
& \left|\left\langle\left[(\ln M) 1_{H}-\ln A\right]\left[\ln A-(\ln m) 1_{H}\right] x, y\right\rangle\right| \\
& \quad \leq \frac{\sqrt{2}}{2 m^{2}}\|y\|(M-m)\langle A x-m x, M x-A x\rangle^{1 / 2} \\
& \quad \leq \frac{\sqrt{2}}{4}\|y\|\|x\|\left(\frac{M}{m}-1\right)^{2}, \\
& \left|\left\langle\left[(\ln M) 1_{H}-\ln A\right]\left[\ln A-(\ln m) 1_{H}\right] x, y\right\rangle\right|  \tag{3.5}\\
& \quad \leq \frac{1}{2}\|y\|\left\langle\left[(\ln M) 1_{H}-\ln A\right]\left[\ln A-(\ln m) 1_{H}\right] x, x\right\rangle^{1 / 2} \ln \left(\frac{M}{m}\right) \\
& \quad \leq \frac{1}{4}\|y\|\|x\|\left[\ln \left(\frac{M}{m}\right)\right]^{2} .
\end{align*}
$$

The proof follows from Theorems 2.4 and 2.5 applied for the logarithmic function.

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