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Research Article

Gronwall-Bellman-Type Integral Inequalities and Applications to BVPs

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We establish some new nonlinear Gronwall-Bellman-Ou-Iang type integral inequalities with two variables. These inequalities generalize former results and can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of differential equations. Example of applying these inequalities to derive the properties of BVPs is also given.

1. Introduction

The Gronwall-Bellman inequality states that if u and f are nonnegative continuous functions on an interval $[a, b]$ satisfying

$$u(t) \leq C + \int_a^t f(s)u(s)ds \tag{1.1}$$

for some constant $C \geq 0$, then

$$u(t) \leq C \exp\left(\int_a^t f(s)ds\right), \quad t \in [a, b]. \tag{1.2}$$

Inequality (1.2) provides an explicit bound to the unknown function and hence furnishes a handy tool in the study of quantitative and qualitative properties of solutions of differential and integral equations. Because of its fundamental importance over the years many generalizations and analogous results of (1.2) have been established (see, e.g., [1–20]). Among various of Gronwall-Bellman-type inequalities, a very useful one is the following.

Theorem 1.1 (see [21]). *If u and f are nonnegative continuous functions defined on $[0,\infty)$ such that*

$$u^2(x) \leq k^2 + 2\int_0^x f(s)u(s)ds \tag{1.3}$$

for all $x \in [0,\infty)$, where $k \geq 0$ is a constant, then

$$u(x) \leq k + \int_0^x f(s)ds \tag{1.4}$$

for all $x \in [0,\infty)$.

Inequality (1.4) is called Ou-Iang's inequality, which was established by Ou-Iang during his study of the boundedness of certain kinds of second-order differential equations.

Recently, Pachpatte established the following generalization of Ou-Iang-type inequality.

Theorem 1.2 (see [20]). *Let u, f, g be nonnegative continuous functions defined on $[0,\infty)$, and let φ be a continuous nondecreasing function on $[0,\infty)$ with $\varphi(r) > 0$ for $r > 0$. If*

$$u^2(x) \leq k^2 + 2\int_0^x [f(s)u(s) + g(s)u(s)\varphi(u(s))]ds \tag{1.5}$$

for all $x \in [0,\infty)$, where $k \geq 0$ is a constant, then

$$u(x) \leq \Phi^{-1}\left[\Phi\left(k + \int_0^x f(s)ds\right) + \int_0^x g(s)ds\right] \tag{1.6}$$

for all $x \in [0, x_1)$, where Φ^{-1} is the inverse function of $\Phi, \Phi(r) := \int_1^r (ds/\varphi(s))$, $r > 0$, and $x_1 \in [0,\infty)$ is chosen such that $\Phi(k + \int_0^x f(s)ds) + \int_0^x g(s)ds \in \mathrm{Dom}(\Phi^{-1})$ for all $x \in [0, x_1)$.

Bainov-Simeonov and Lipovan gave the following Gronwall-Bellman-type inequalities, which are useful in the study of global existence of solutions of certain integral equations and functional differential equations.

Theorem 1.3 (see [1]). *Let $I = [0,a]$, $J = [0,b]$, where, for the sake of covenience, we allow a, b to be $+\infty$ (in this case we mean interval $[0,\infty)$). Let $c \geq 0$ be a constant, let $\varphi \in C([0,\infty),[0,\infty))$ be nondecreasing with $\varphi(r) > 0$ for $r > 0$, and $b \in C(I \times J, [0,\infty))$. If $u \in C(I \times J, [0,\infty))$ satisfies*

$$u(x,y) \leq c + \int_0^x \int_0^y b(s,t)\varphi(u(s,t))dt\,ds \tag{1.7}$$

for all $(x,y) \in I \times J$, then

$$u(x,y) \leq \Phi^{-1}\left[\Phi(c) + \int_0^x \int_0^y b(s,t)dt\,ds\right] \tag{1.8}$$

for all $(x, y) \in [0, x_1] \times [0, y_1]$*, where* Φ *is defined as in Theorem 1.2 and* $(x_1, y_1) \in I \times J$ *is chosen such that* $\Phi(c) + \int_0^x \int_0^y b(s,t)dt\,ds \in \mathrm{Dom}(\Phi^{-1})$ *for all* $(x, y) \in [0, x_1] \times [0, y_1]$.

Theorem 1.4 (see [13]). *Suppose* u, f *are nonnegative continuous functions defined on* $[x_o, X)$, $\varphi \in C[0,\infty)$ *with* $\varphi(r) > 0$ *for* $r > 0$*, and* $\alpha \in C^1([x_o, X), [x_o, X))$ *with* $\alpha(x) \le x$ *on* $[x_o, X)$ *are nondecreasing functions. If*

$$u(x) \le k + \int_{\alpha(x_o)}^{\alpha(x)} f(s)\varphi(u(s))ds \tag{1.9}$$

for all $x \in [x_o, X)$*, where* $k \ge 0$ *is a constant, then*

$$u(x) \le \Phi^{-1}\left[\Phi(k) + \int_{\alpha(x_o)}^{\alpha(x)} f(s)ds\right] \tag{1.10}$$

for all $x \in [x_o, x_1)$*, where* Φ *is defined as in Theorem 1.2, and* $x_1 \in [x_o, X)$ *is chosen such that* $\Phi(k) + \int_{\alpha(x_o)}^{\alpha(x)} f(s)ds \in \mathrm{Dom}(\Phi^{-1})$ *for all* $x \in [x_o, x_1)$.

Very recently, the above results have been further generalized by Cheung to the following.

Theorem 1.5 (see [7]). *Let* $k \ge 0$ *and* $p > q > 0$ *be constants. Let* $a, b \in C([x_o, X) \times [y_o, Y), [0, +\infty))$, $\alpha, \gamma \in C^1([x_o, X), [x_o, X))$, $\beta, \delta \in C^1([y_o, Y), [y_o, Y))$*, and* $\varphi, h \in C([0, +\infty), [0, +\infty))$ *be functions satisfying*

(i) $\alpha, \beta, \gamma, \delta$ *are nondecreasing and* $\alpha, \gamma \le \mathrm{id}_I$, $\beta, \delta \le \mathrm{id}_J$;

(ii) φ *is nondecreasing with* $\varphi(r) > 0$ *for* $r > 0$.

If $u \in C([x_o, X) \times [y_o, Y), [0, +\infty))$ *satisfies*

$$\begin{aligned} u^p(x,y) \le k &+ \frac{p}{p-q}\int_{\alpha(x_o)}^{\alpha(x)}\int_{\beta(y_o)}^{\beta(y)} a(s,t)u^q(s,t)dt\,ds \\ &+ \frac{p}{p-q}\int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)u^q(s,t)\varphi(u(s,t))dt\,ds \end{aligned} \tag{1.11}$$

for all $(x, y) \in [x_o, X) \times [y_o, Y)$*, then*

$$u(x,y) \le \left\{\Phi_{p-q}^{-1}\left[\Phi_{p-q}\left(k^{1-q/p} + A(x,y)\right) + B(x,y)\right]\right\}^{1/(p-q)} \tag{1.12}$$

for all $(x, y) \in [x_o, x_1] \times [y_o, y_1]$*, where*

$$A(x,y) = \int_{\alpha(x_o)}^{\alpha(x)}\int_{\beta(y_o)}^{\beta(y)} a(s,t)dt\,ds, \qquad B(x,y) = \int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)dt\,ds, \tag{1.13}$$

Φ_{p-q}^{-1} is the inverse function of $\Phi_{p-q}, \Phi_{p-q}(r) := \int_1^r (ds/\varphi(s^{1/(p-q)})) \in C([0,+\infty),[0,+\infty))$, and $(x_1,y_1) \in [x_o, X) \times [y_o, Y)$ is chosen such that $\Phi_{p-q}(k^{1-q/p} + A(x,y)) + B(x,y) \in \mathrm{Dom}(\Phi_{p-q}^{-1})$ for all $(x,y) \in [x_o,x_1] \times [y_o,y_1]$.

In this paper, we intend to establish some new nonlinear Gronwall-Bellman-Ou-Iang type integral inequalities with two variables. The setup is basically along the line of [7] but this is by all means a nontrivial improvement of the results there. Examples are also given to illustrate the usefulness of these inequalities in the study of qualitative as well as the quantitative properties of solutions of BVPs.

2. Main Results

Throughout this paper, $x_o, y_o \in \mathbb{R}$ are two fixed numbers. Let $I = [x_o, X) \subset \mathbb{R}$, $J = [y_o, Y) \subset \mathbb{R}$, and $\Delta = I \times J \subset \mathbb{R}^2$, here we allow X or Y to be $+\infty$. We denote by $C^i(U,V)$ the set of all i-times continuously differentiable functions of U into V, and $C^o(U,V) = C(U,V)$. Partial derivatives of a function $z(x,y)$ are denoted by z_x, z_y, z_{xy}, and so forth. The identity function will be denoted as id and so, in particular, id_U is the identity function of U onto itself.

Let $\mathbb{R}_+ = [0,+\infty)$ and for any $\varphi, \psi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$, define

$$\begin{gathered}\Phi_h(r) := \int_1^r \frac{ds}{\varphi(h^{-1}(s))}, \quad \Psi_h(r) := \int_1^r \frac{ds}{\psi(h^{-1}(s))}, \quad r > 0, \\ \Phi_h(0) := \lim_{r\to 0^+} \Phi_h(r), \qquad \Psi_h(0) := \lim_{r\to 0^+} \Psi_h(r).\end{gathered} \tag{2.1}$$

Note that we allow $\Phi_h(0)$ and $\Psi_h(0)$ to be $-\infty$ here.

Theorem 2.1. *Suppose $u \in C(\Delta, \mathbb{R}_+)$. Let $c \geq 0$ be a constant. If $\gamma \in C^1(I,I)$, $\delta \in C^1(J,J)$, $b \in C(\Delta, \mathbb{R}_+)$, and $\varphi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *γ, δ are nondecreasing and $\gamma \leq \mathrm{id}_I$, $\delta \leq \mathrm{id}_J$;*

(ii) *φ is nondecreasing with $\varphi(r) > 0$ for $r > 0$;*

(iii) *h is strictly increasing with $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$;*

(iv) *for any $(x,y) \in \Delta$,*

$$h(u(x,y)) \leq c + \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t)\varphi(u(s,t))dt\,ds, \tag{2.2}$$

then we have

$$u(x,y) \leq h^{-1}\{\Phi_h^{-1}[\Phi_h(c) + B(x,y)]\} \tag{2.3}$$

for all $(x,y) \in [x_o,x_1] \times [y_o,y_1]$, where

$$B(x,y) = \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t)dt\,ds, \tag{2.4}$$

Φ_h^{-1} is the inverse function of Φ_h, and $(x_1, y_1) \in \Delta$ is chosen such that $\Phi_h(c) + B(x, y) \in \mathrm{Dom}(\Phi_h^{-1})$ for all $(x, y) \in [x_o, x_1] \times [y_o, y_1]$.

Proof. It suffices to only consider the case $c > 0$, since the case $c = 0$ can then be arrived at by continuity argument. If we let $g(x, y)$ denote the right-hand side of (2.2), then we have $g > 0$, $u \le h^{-1}(g)$ on Δ, and g is nondecreasing in each variable. Hence for any $(x, y) \in \Delta$,

$$\begin{aligned} g_x(x,y) &= \gamma'(x)\int_{\delta(y_o)}^{\delta(y)} b(\gamma(x),t)\varphi(u(\gamma(x),t))dt \\ &\le \gamma'(x)\int_{\delta(y_o)}^{\delta(y)} b(\gamma(x),t)\varphi(h^{-1}(g(\gamma(x),t)))dt \\ &\le \gamma'(x)\varphi(h^{-1}(g(\gamma(x),\delta(y))))\int_{\delta(y_o)}^{\delta(y)} b(\gamma(x),t)dt \\ &\le \gamma'(x)\varphi(h^{-1}(g(x,y)))\int_{\delta(y_o)}^{\delta(y)} b(\gamma(x),t)dt. \end{aligned} \tag{2.5}$$

By the definition of Φ_h,

$$\begin{aligned} (\Phi_h \circ g)_x(x,y) &= \left.\frac{d\Phi_h}{dr}\right|_{g(x,y)} \cdot g_x(x,y) \\ &\le \frac{1}{\varphi(h^{-1}(g(x,y)))} \cdot \gamma'(x)\varphi(h^{-1}(g(x,y)))\int_{\delta(y_o)}^{\delta(y)} b((\gamma(x),t))dt \\ &\le \gamma'(x)\left(\int_{\delta(y_o)}^{\delta(y)} b(\gamma(x),t)dt\right). \end{aligned} \tag{2.6}$$

Integrating with respect to x over $[x_o, x]$, we have

$$\begin{aligned} \Phi_h(g(x,y)) - \Phi_h(g(x_o,y)) &\le \int_{x_o}^{x} \gamma'(\xi)\left(\int_{\delta(y_o)}^{\delta(y)} b(\gamma(\xi),t)dt\right)d\xi \\ &\le \int_{\gamma(x_o)}^{\gamma(x_1)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)dt\,ds, \end{aligned} \tag{2.7}$$

that is,

$$\Phi_h(g(x,y)) \le \Phi_h(g(x_o,y)) + B(x,y). \tag{2.8}$$

Therefore,

$$u(x,y) \le h^{-1}(g) \le h^{-1}\{\Phi_h^{-1}[\Phi_h(c) + B(x,y)]\} \tag{2.9}$$

for all $(x, y) \in [x_o, x_1] \times [y_o, y_1]$. □

Corollary 2.2. *Suppose* $u \in C(\Delta, \mathbb{R}_+)$. *Let* $c \geq 0$ *be a constant. If* $\gamma \in C^1(I, I)$, $\delta \in C^1(J, J)$, $b \in C(\Delta, \mathbb{R}_+)$, *and* $h \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) γ, δ *are nondecreasing and* $\gamma \leq \mathrm{id}_I$, $\delta \leq \mathrm{id}_J$;

(ii) h *is strictly increasing with* $h(0) = 0$ *and* $h(t) \to \infty$ *as* $t \to \infty$;

(iii) *for any* $(x, y) \in \Delta$,

$$h(u(x,y)) \leq c + \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t) h(u(s,t)) dt\, ds, \tag{2.10}$$

then we have

$$u(x,y) \leq h^{-1}\left(c e^{B(x,y)}\right) \tag{2.11}$$

for all $(x, y) \in \Delta$, *where*

$$B(x,y) = \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t) dt\, ds. \tag{2.12}$$

Proof. Let $\varphi = h$, then $\Phi_h(r) = \ln r$. The corollary now follows immediately from Theorem 2.1. □

Remark 2.3. It is easily seen that Theorem 2.1 generalizes Theorems 1.3 and 1.4.

Theorem 2.4. *Suppose* $u \in C(\Delta, \mathbb{R}_+)$. *Let* $c \geq 0$ *be a constant. If* $\alpha, \gamma \in C^1(I, I)$, $\beta, \delta \in C^1(J, J)$, $a, b \in C(\Delta, \mathbb{R}_+)$, *and* $\varphi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ *are functions satisfying*

(i) $\alpha, \beta, \gamma, \delta$ *are nondecreasing and* $\alpha, \gamma \leq \mathrm{id}_I$, $\beta, \delta \leq \mathrm{id}_J$;

(ii) φ *is nondecreasing with* $\varphi(r) > 0$ *for* $r > 0$;

(iii) $h(t)$ *and* $H(t) := h(t)/t$, $t > 0$, *are strictly increasing with* $H(t) \to +\infty$ *as* $t \to +\infty$;

(iv) *for any* $(x, y) \in \Delta$,

$$h(u(x,y)) \leq c + \int_{\alpha(x_o)}^{\alpha(x)} \int_{\beta(y_o)}^{\beta(y)} a(s,t) u(s,t) dt\, ds + \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t) u(s,t) \varphi(u(s,t)) dt\, ds, \tag{2.13}$$

then we have

$$u(x,y) \leq H^{-1}\left\{\Phi_H^{-1}\left[\Phi_H\left(\frac{c}{h^{-1}(c)} + A(x,y)\right) + B(x,y)\right]\right\} \tag{2.14}$$

for all $(x, y) \in [x_o, x_1] \times [y_o, y_1]$, *where*

$$A(x,y) = \int_{\alpha(x_o)}^{\alpha(x)} \int_{\beta(y_o)}^{\beta(y)} a(s,t) dt\, ds, \qquad B(x,y) = \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t) dt\, ds, \tag{2.15}$$

Φ_H^{-1} *is the inverse function of* Φ_H, *and* $(x_1, y_1) \in \Delta$ *is chosen such that* $\Phi_H(c/h^{-1}(c) + A(x, y)) + B(x, y) \in \mathrm{Dom}(\Phi_H^{-1})$ *for all* $(x, y) \in [x_o, x_1] \times [y_o, y_1]$.

Proof. It suffices to consider the case $c > 0$. If we let $g(x, y)$ denote the right-hand side of (2.13), then we have $g > 0$, $u \le h^{-1}(g)$ on Δ, and g is nondecreasing in each variable. Hence for any $(x, y) \in \Delta$,

$$
\begin{aligned}
g_x(x, y) &= \alpha'(x) \int_{\beta(y_o)}^{\beta(y)} a(\alpha(x), t) u(\alpha(x), t) dt \\
&\quad + \gamma'(x) \int_{\delta(y_o)}^{\delta(y)} b(\gamma(x), t) u(\gamma(x), t) \varphi(u(\gamma(x), t)) dt \\
&\le \alpha'(x) \int_{\beta(y_o)}^{\beta(y)} a(\alpha(x), t) h^{-1}(g(\alpha(x), t)) dt \\
&\quad + \gamma'(x) \int_{\delta(y_o)}^{\delta(y)} b(\gamma(x), t) h^{-1}(g(\gamma(x), t)) \varphi(h^{-1}(g(\gamma(x), t))) dt \\
&\le \alpha'(x) h^{-1}(g(\alpha(x), \beta(y))) \int_{\beta(y_o)}^{\beta(y)} a(\alpha(x), t) dt \\
&\quad + \gamma'(x) h^{-1}(g(\gamma(x), \beta(y))) \int_{\delta(y_o)}^{\delta(y)} b(\gamma(x), t) \varphi(h^{-1}(g(\gamma(x), t))) dt \\
&\le \alpha'(x) h^{-1}(g(x, y)) \int_{\beta(y_o)}^{\beta(y)} a(\alpha(x), t) dt \\
&\quad + \gamma'(x) h^{-1}(g(x, y)) \int_{\delta(y_o)}^{\delta(y)} b(\gamma(x), t) \varphi(h^{-1}(g(\gamma(x), t))) dt,
\end{aligned}
\tag{2.16}
$$

that is,

$$
\frac{g_x(x, y)}{h^{-1}(g(x, y))} \le \alpha'(x) \int_{\beta(y_o)}^{\beta(y)} a(\alpha(x), t) dt + \gamma'(x) \int_{\delta(y_o)}^{\delta(y)} b(\gamma(x), t) \varphi(h^{-1}(g(\gamma(x), t))) dt. \tag{2.17}
$$

Integrating with respect to x over $[x_o, x]$, we get

$$
\begin{aligned}
\int_{x_o}^{x} \frac{g_x(\xi, y)}{h^{-1}(g(\xi, y))} d\xi &\le \int_{x_o}^{x} \alpha'(\xi) \left(\int_{\beta(y_o)}^{\beta(y)} a(\alpha(\xi), t) dt \right) d\xi \\
&\quad + \int_{x_o}^{x} \gamma'(\xi) \left(\int_{\delta(y_o)}^{\delta(y)} b(\gamma(\xi), t) \varphi(h^{-1}(g(\gamma(\xi), t))) dt \right) d\xi \\
&\le \int_{\alpha(x_o)}^{\alpha(x)} \int_{\beta(y_o)}^{\beta(y)} a(s, t) dt\, ds \\
&\quad + \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s, t) \varphi(h^{-1}(g(s, t))) dt\, ds \\
&= A(x, y) + \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s, t) \varphi(h^{-1}(g(s, t))) dt\, ds.
\end{aligned}
\tag{2.18}
$$

Since

$$\int_{x_o}^{x} \frac{g_\xi(\xi,y)}{h^{-1}(g(\xi,y))}d\xi = \frac{g(\xi,y)}{h^{-1}(g(\xi,y))}\bigg|_{x_o}^{x} + \int_{x_o}^{x} g(\xi,y)\frac{h_g^{-1}\cdot g_\xi(\xi,y)}{\left(h^{-1}(g(\xi,y))\right)^2}d\xi, \tag{2.19}$$

h is strictly increasing and $g > 0$ is nondecreasing with respect to each variable. Together with (2.18), we have

$$\begin{aligned}\int_{x_o}^{x} \frac{g_x(x,y)}{h^{-1}(g(x,y))} &\geq \frac{g(x,y)}{h^{-1}(g(x,y))}\bigg|_{x_o}^{x} \\ &= \frac{g(x,y)}{h^{-1}(g(x,y))} - \frac{g(x_o,y)}{h^{-1}(g(x_o,y))} \\ &= H\left(h^{-1}(g(x,y))\right) - \frac{c}{h^{-1}(c)}\end{aligned} \tag{2.20}$$

for all $(x,y) \in \Delta$. Hence,

$$H\left(h^{-1}(g(x,y))\right) \leq \frac{c}{h^{-1}(c)} + A(x,y) + \int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)\varphi\left(h^{-1}(g(s,t))\right)dt\,ds \tag{2.21}$$

for all $(x,y) \in \Delta$.

For any fixed $(\overline{x},\overline{y}) \in [x_o,x_1] \times [y_o,y_1]$, by the fact that A is nondecreasing in each variable, we have

$$H\left(h^{-1}(g(x,y))\right) \leq \left(\frac{c}{h^{-1}(c)} + A(\overline{x},\overline{y})\right) + \int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)\varphi\left(h^{-1}(g(s,t))\right)dt\,ds \tag{2.22}$$

for all $(x,y) \in [x_o,\overline{x}] \times [y_o,\overline{y}]$. Now by applying Theorem 2.1 to the strictly increasing function H, we have

$$h^{-1}(g(x,y)) \leq H^{-1}\left\{\Phi_H^{-1}\left[\Phi_H\left(\frac{c}{h^{-1}(c)} + A(\overline{x},\overline{y})\right) + B(x,y)\right]\right\} \tag{2.23}$$

for all $(x,y) \in [x_o,\overline{x}] \times [y_o,\overline{y}]$. In particular, this leads to

$$u(\overline{x},\overline{y}) \leq h^{-1}(g(\overline{x},\overline{y})) \leq H^{-1}\left\{\Phi_H^{-1}\left[\Phi_H\left(\frac{c}{h^{-1}(c)} + A(\overline{x},\overline{y})\right) + B(\overline{x},\overline{y})\right]\right\}. \tag{2.24}$$

Since $(\overline{x},\overline{y}) \in [x_o,x_1] \times [y_o,y_1]$ is arbitrary, this concludes the proof of the theorem. □

Corollary 2.5. *Suppose $u \in C(I, \mathbb{R}_+)$. Let $c \geq 0$ be a constant. If $\alpha, \gamma \in C^1(I, I)$, $a, b \in C(I, \mathbb{R}_+)$, and $\varphi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *α, γ are nondecreasing and $\alpha, \gamma \leq \mathrm{id}_I$;*

(ii) *φ is nondecreasing with $\varphi(r) > 0$ for $r > 0$;*

(iii) *$h(t)$ and $H(t) := h(t)/t$, $t > 0$, are strictly increasing with $H(t) \to +\infty$ as $t \to +\infty$;*

(iv) *for any $x \in I$,*

$$h(u(x,y)) \leq c + \int_{\alpha(x_o)}^{\alpha(x)} a(s)u(s)ds + \int_{\gamma(x_o)}^{\gamma(x)} b(s)u(s)\varphi(u(s))ds, \tag{2.25}$$

then we have

$$u(x,y) \leq H^{-1}\left\{\Phi_H^{-1}\left[\Phi_H\left(\frac{c}{h^{-1}(c)} + A(x)\right) + B(x)\right]\right\} \tag{2.26}$$

for all $x \in [x_o, x_1]$, where

$$A(x) = \int_{\alpha(x_o)}^{\alpha(x)} a(s)ds, \qquad B(x) = \int_{\gamma(x_o)}^{\gamma(x)} b(s)ds, \tag{2.27}$$

Φ_H^{-1} is the inverse function of Φ_H, and $x_1 \in I$ is chosen such that $\Phi_H(c/h^{-1}(c) + A(x)) + B(x) \in \mathrm{Dom}(\Phi_H^{-1})$ for all $x \in [x_o, x_1]$.

Remark 2.6. If we choose $h(t) = t^2$, $c = k^2$, $\alpha = \gamma = \mathrm{id}_I$, $a(s) = f(s)$, and $b(s) = g(s)$, then Corollary 2.5 reduces to Theorem 1.2.

Theorem 2.7. *Suppose $u \in C(\Delta, \mathbb{R}_+)$. Let $c \geq 0$ be a constant. If $\alpha, \gamma \in C^1(I, I)$, $\beta, \delta \in C^1(J, J)$, $a, b \in C(\Delta, \mathbb{R}_+)$, and $\varphi, h \in C(\mathbb{R}_+, \mathbb{R}_+)$ are functions satisfying*

(i) *$\alpha, \beta, \gamma, \delta$ are nondecreasing and $\alpha, \gamma \leq \mathrm{id}_I$, $\beta, \delta \leq \mathrm{id}_J$;*

(ii) *φ is nondecreasing with $\varphi(r) > 0$ for $r > 0$;*

(iii) *h, Q, $H(t) := h(t)/t$, $\overline{H}(t) := h \circ Q^{-1}$ and $\widetilde{H} := h \circ Q^{-1}/t$, $t > 0$, are strictly increasing with $Q(r) > 0$ when $r > 0$ and $\widetilde{H}(t) \to +\infty$ as $t \to +\infty$;*

(iv) *for any $(x, y) \in \Delta$,*

$$\begin{aligned} h(u(x,y)) \leq c &+ \int_{\alpha(x_o)}^{\alpha(x)} \int_{\beta(y_o)}^{\beta(y)} a(s,t)Q(u(s,t))dt\,ds \\ &+ \int_{\gamma(x_o)}^{\gamma(x)} \int_{\delta(y_o)}^{\delta(y)} b(s,t)Q(u(s,t))\varphi(u(s,t))dt\,ds, \end{aligned} \tag{2.28}$$

then we have

$$u(x,y) \le Q^{-1} \circ \widetilde{H}^{-1}\left\{\Psi_{\widetilde{H}}^{-1}\left[\Psi_{\widetilde{H}}\left(\frac{c}{\overline{H}^{-1}(c)} + A(x,y)\right) + B(x,y)\right]\right\} \tag{2.29}$$

for all $(x,y) \in [x_o, x_1] \times [y_o, y_1]$, *where* $A(x,y), B(x,y)$ *are the same as in Theorem 2.4,* $\psi := \varphi \circ Q^{-1}$, *and* $(x_1, y_1) \in \Delta$ *is chosen such that* $\Psi_{\widetilde{H}}(c/\overline{H}^{-1}(c) + A(x,y)) + B(x,y) \in \mathrm{Dom}(\Psi_{\widetilde{H}}^{-1})$ *for all* $(x,y) \in [x_o, x_1] \times [y_o, y_1]$.

Proof. It suffices to consider only the case $c > 0$. For any $r > 0$, define $\psi(r) := \varphi(Q^{-1}(r))$. Then clearly ψ satisfies condition (ii) of Theorem 2.4. Let $v = Q(u)$, from (2.18), we have

$$\begin{aligned} h(Q^{-1}(v(x,y))) \le c &+ \int_{\alpha(x_o)}^{\alpha(x)}\int_{\beta(y_o)}^{\beta(y)} a(s,t)v(s,t)\,dt\,ds \\ &+ \int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)v(s,t)\varphi(Q^{-1}(u(s,t)))\,dt\,ds \end{aligned} \tag{2.30}$$

or

$$\begin{aligned} \overline{H}(v(x,y)) \le c &+ \int_{\alpha(x_o)}^{\alpha(x)}\int_{\beta(y_o)}^{\beta(y)} a(s,t)v(s,t)\,dt\,ds \\ &+ \int_{\gamma(x_o)}^{\gamma(x)}\int_{\delta(y_o)}^{\delta(y)} b(s,t)v(s,t)\psi(u(s,t))\,dt\,ds. \end{aligned} \tag{2.31}$$

From Theorem 2.4, we have

$$v(x,y) \le \widetilde{H}^{-1}\left\{\Psi_{\widetilde{H}}^{-1}\left[\Psi_{\widetilde{H}}\left(\frac{c}{\overline{H}^{-1}(c)} + A(x,y)\right) + B(x,y)\right]\right\}, \tag{2.32}$$

that is,

$$u(x,y) \le Q^{-1} \circ \widetilde{H}^{-1}\left\{\Psi_{\widetilde{H}}^{-1}\left[\Psi_{\widetilde{H}}\left(\frac{c}{\overline{H}^{-1}(c)} + A(x,y)\right) + B(x,y)\right]\right\} \tag{2.33}$$

for all $(x,y) \in [x_o, x_1] \times [y_o, y_1]$. □

Remark 2.8. Theorem 2.7 is a generalization of Theorem 1.5.

3. Application to Boundary Value Problems

In this section, we use the results obtained in Section 2 to study certain properties of positive solutions of the following boundary value problem (BVP):

$$\begin{gathered}[h(u(x,y))]_{xy} = F(x,y,u(\alpha(x),\gamma(y))),\\ u(x,y_o) = f(x), \quad u(x_o,y) = g(y), \quad f(x_o) = g(y_o) = 0,\end{gathered} \tag{3.1}$$

where h is defined as in Theorem 2.1, $F \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $f \in C(I, \mathbb{R})$, and $g \in C(I, \mathbb{R})$ are given.

Our first result deals with the boundedness of solutions.

Theorem 3.1. *Consider BVP* (3.1). *If*

(i) $|F(x,y,u)| \leq b(x,y)\varphi(|u|)$ *for some* $b \in C(\Delta, \mathbb{R}_+)$;

(ii) $|h(f(x)) + h(g(y))| \leq K$ *for some* $K \geq 0$,

then all positive solutions to BVP (3.1) *satisfy*

$$u(x,y) \leq h^{-1}\{\Phi_h^{-1}[\Phi(K) + B(x,y)]\}, \quad (x,y) \in \Delta, \tag{3.2}$$

where φ, Φ_h, *and* Φ_h^{-1} *are defined as in Theorem 2.1, and*

$$B(x,y) = \int_{x_o}^{x}\int_{y_o}^{y} b(s,t)dt\,ds. \tag{3.3}$$

In particular, if $B(x,y)$ *is bounded on* Δ, *then every solution to BVP* (3.1) *is bounded on* Δ.

Proof. It is easily seen that $u = u(x,y)$ solves BVP (3.1) if and only if it satisfies the integral equation

$$h(u(x,y)) = h(f(x)) + h(g(y)) + \int_{x_o}^{x}\int_{y_o}^{y} F(s,t,u(\alpha(x),\gamma(y)))dt\,ds. \tag{3.4}$$

Hence by (i) and (ii),

$$|h(u(x,y))| \leq K + \int_{x_o}^{x}\int_{y_o}^{y} b(s,t)|\varphi(u(\alpha(s),\gamma(t)))|dt\,ds. \tag{3.5}$$

Changing variables by letting $\sigma = \alpha(s)$, $\tau = \gamma(t)$, we get

$$|h(u(x,y))| \leq K + \int_{\alpha(x_o)}^{\alpha(x)}\int_{\gamma(y_o)}^{\gamma(y)} b(\alpha^{-1}(\sigma),\gamma^{-1}(\tau))|\varphi(u(\sigma,\tau))|(\alpha^{-1})'(\sigma)(\gamma^{-1})'(\tau)d\tau\,d\sigma. \tag{3.6}$$

From Theorem 2.1, we have

$$u(x,y) \le h^{-1}\left\{\Phi_h^{-1}\left[\Phi_h(K)+\int_{\alpha(x_o)}^{\alpha(x)}\int_{\gamma(y_o)}^{\gamma(y)} b\left(\alpha^{-1}(\sigma),\gamma^{-1}(\tau)\right)\left(\alpha^{-1}\right)'(\sigma)\left(\gamma^{-1}\right)'(\tau)d\tau\, d\sigma\right]\right\}, \quad (3.7)$$

that is,

$$u(x,y) \le h^{-1}\left\{\Phi_h^{-1}\left[\Phi_h(K)+B(x,y)\right]\right\}. \quad (3.8)$$

□

The next result is about the quantitative property of solutions.

Theorem 3.2. *Consider BVP* (3.1). *If*

$$\left|F(x,y,u_1)-F(x,y,u_2)\right| \le b(x,y)\left|h(u_1)-h(u_2)\right| \quad (3.9)$$

for some $b \in C(\Delta,\mathbb{R}_+)$, then (BVP) (3.1) *has at most one solution on* Δ.

Proof. Assume that $u_1(x,y)$ and $u_2(x,y)$ are two solutions to BVP (3.1). By (3.4), we have

$$\begin{aligned}\left|h(u_1(x,y))-h(u_2(x,y))\right| &\le \int_{x_o}^{x}\int_{y_o}^{y}\left|F_1(s,t,u(\alpha(x),\gamma(y)))-F_2(s,t,u(\alpha(x),\gamma(y)))\right|dt\,ds\\ &\le \int_{x_o}^{x}\int_{y_o}^{y} b(s,t)\left|h(u_1(\alpha(s),\gamma(t)))-h(u_2(\alpha(s),\gamma(t)))\right|dt\,ds.\end{aligned}$$

Changing variables by letting $\sigma=\alpha(s)$, $\tau=\gamma(t)$, we get

$$\begin{aligned}&\left|h(u_1(x,y))-h(u_2(x,y))\right|\\ &\quad\le \int_{\alpha(x_o)}^{\alpha(x)}\int_{\gamma(y_o)}^{\gamma(y)} b\left(\alpha^{-1}(\sigma),\gamma^{-1}(\tau)\right)\left|h(u_1(\sigma,\tau))-h(u_2(\sigma,\tau))\right|\left(\alpha^{-1}\right)'(\sigma)\left(\gamma^{-1}\right)'(\tau)d\tau\ d\sigma.\end{aligned} \quad (3.10)$$

From Corollary 2.2, we have

$$\left|h(u_1(x,y))-h(u_2(x,y))\right| \le 0, \quad (3.11)$$

that is,

$$u_1(x,y)=u_2(x,y), \quad \forall (x,y)\in\Delta. \quad (3.12)$$

□

Finally, we investigate the continuous dependence of the solutions of BVP (3.1) on the functional F and the boundary data f and g. For this, we consider the following variation of

BVP (3.1):

$$\begin{gathered}[h(u(x,y))]_{xy} = \overline{F}(x,y,u(\alpha(x),\gamma(y))),\\ u(x,y_o) = \overline{f}(x), \quad u(x_o,y) = \overline{g}(y), \quad \overline{f}(x_o) = \overline{g}(y_o) = 0,\end{gathered} \tag{3.13}$$

where h is defined as in Theorem 2.1, $\overline{F} \in C(\Delta \times \mathbb{R}, \mathbb{R})$, $\overline{f} \in C(I,\mathbb{R})$, and $\overline{g} \in C(I,\mathbb{R})$ are given.

Theorem 3.3. *Consider BVP* (3.1) *and BVP* (3.13)*. If*

(i) $|F(x,y,u) - F(x,y,\overline{u})| \leq b(x,y)|h(u) - h(\overline{u})|$ *for some* $b \in C(\Delta, \mathbb{R}_+)$*;*

(ii) $|(h(f(x)) - h(\overline{f}(x))) + (h(g(x)) - h(\overline{g}(x)))| \leq \epsilon/2$*;*

(iii) *for all solutions* $\overline{u}(x,y)$ *of BVP* (3.13)*,*

$$\int_{x_o}^{x}\int_{y_o}^{y} \left|F(s,t,u(\alpha(x),\gamma(y))) - \overline{F}(s,t,\overline{u}(\alpha(x),\gamma(y)))\right| dt\, ds \leq \frac{\epsilon}{2}, \tag{3.14}$$

then

$$\left|h(u(x,y)) - \overline{h}(\overline{u}(x,y))\right| \leq \epsilon e^{B(x,y)}, \tag{3.15}$$

where $B(x,y)$ *is as defined in Theorem 3.1. Hence* $h(u(x,y))$ *depends continuously on* F, f*, and* g*.*

Proof. Let $u(x,y)$ and $\overline{u}(x,y)$ be solutions to BVP (3.1) and BVP (3.13), respectively. Then

$$\begin{aligned} h(u(x,y)) &= h(f(x)) + h(g(y)) + \int_{x_o}^{x}\int_{y_o}^{y} F(s,t,u(\alpha(x),\gamma(y)))dt\, ds,\\ h(\overline{u}(x,y)) &= h(\overline{f}(x)) + h(\overline{g}(y)) + \int_{x_o}^{x}\int_{y_o}^{y} \overline{F}(s,t,\overline{u}(\alpha(x),\gamma(y)))dt\, ds.\end{aligned} \tag{3.16}$$

Hence from assumption (ii), we have

$$\begin{aligned} &\left|h(u(x,y)) - \overline{h}(\overline{u}(x,y))\right|\\ &\quad\leq \left[\left|(h(f(x)) - \overline{h}(f(x))) + (h(g(y)) - \overline{h}(g(y)))\right|\right]\\ &\qquad+ \int_{x_o}^{x}\int_{y_o}^{y} \left|F(s,t,u(\alpha(x),\gamma(y))) - \overline{F}(s,t,\overline{u}(\alpha(x),\gamma(y)))\right| dt\, ds\\ &\quad\leq \frac{\epsilon}{2} + \int_{x_o}^{x}\int_{y_o}^{y} \left|F(s,t,u(\alpha(x),\gamma(y))) - F(s,t,\overline{u}(\alpha(x),\gamma(y)))\right| dt\, ds\\ &\qquad+ \int_{x_o}^{x}\int_{y_o}^{y} \left|F(s,t,\overline{u}(\alpha(x),\gamma(y))) - \overline{F}(s,t,\overline{u}(\alpha(x),\gamma(y)))\right| dt\, ds\\ &\quad\leq \epsilon + \int_{x_o}^{x}\int_{y_o}^{y} b(x,y)\left|h(u(x,y)) - h(\overline{u}(x,y))\right| dt\, ds.\end{aligned} \tag{3.17}$$

From Corollary 2.2, we have

$$|h(u(x,y)) - \overline{h}(\overline{u}(x,y))| \leq \epsilon e^{B(x,y)}. \tag{3.18}$$

If we restrict to any compact subset A of Δ, then $B(x,y)$ is bounded, and hence

$$\left|h(u(x,y)) - \overline{h}(\overline{u}(x,y))\right| \leq \epsilon K \tag{3.19}$$

for some $K > 0$ for all $(x,y) \in A$. Therefore, $h(u(x,y))$ depends continuously on F, f, and g. □

Remark 3.4. The uniqueness of solution is often a direct consequence of the continuous dependence on parameters. In fact, let BVP (3.13) be coincide with (3.1), then according to Theorem 3.3,

$$\left|h(u(x,y)) - \overline{h}(\overline{u}(x,y))\right| \leq \epsilon e^{B(x,y)} \quad \forall \epsilon > 0, \tag{3.20}$$

thus $u(x,y) = \overline{u}(x,y)$, and so Theorem 3.2 can be viewed as a corollary of Theorem 3.3.

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