

Research Article

On the Stability of Cubic Mappings and Quadratic Mappings in Random Normed Spaces

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Recently, the stability of the cubic functional equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$ in fuzzy normed spaces was proved in earlier work; and the stability of the additive functional equations $f(x + y) = f(x) + f(y)$, $2f((x + y)/2) = f(x) + f(y)$ in random normed spaces was proved as well. In this paper, we prove the stability of the cubic functional equation $f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x)$ in random normed spaces by an alternative proof which provides a better estimation. Finally, we prove the stability of the quartic functional equation $f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y)$ in random normed spaces.

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1. Introduction and preliminaries

The study of stability problems for functional equations is related to a question of Ulam [1] concerning the stability of group homomorphisms and affirmatively answered for Banach spaces by Hyers [2]. Subsequently, the result of Hyers was generalized by Aoki [3] for additive mappings and by Th. M. Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Th. M. Rassias has provided a lot of influence in the development of what we now call Hyers-Ulam-Rassias stability of functional equations. We refer the interested readers for more information on such problems to the papers [5–9]. The functional equation

$$f(2x + y) + f(2x - y) = 2f(x + y) + 2f(x - y) + 12f(x) \quad (1.1)$$

is said to be the cubic functional equation since the function $f(x) = cx^3$ is its solution. Every solution of the cubic functional equation is said to be a cubic mapping. The stability problem for the cubic functional equation was proved by Jun and Kim [10] for mappings $f : X \rightarrow Y$, where X is a real normed space and Y is a Banach space. Later, a number of mathematicians have worked on the stability of some types of the cubic equation [11]. The functional equation

$$f(2x + y) + f(2x - y) = 4f(x + y) + 4f(x - y) + 24f(x) - 6f(y) \quad (1.2)$$

is said to be the quadratic functional equation since the function $f(x) = cx^4$ is its solution. Every solution of the quadratic functional equation is said to be a quadratic mapping. The stability problem for the quadratic functional equation first was proved by J. M. Rassias [12] for mappings $f : X \rightarrow Y$, where X is a real normed space and Y is a Banach space. In addition, Mirmostafae et al. [13–15], Alsina [16], Miheţ and Radu [17] investigated the stability in the settings of fuzzy, probabilistic, and random normed spaces.

In the sequel, we will adopt the usual terminology, notations, and conventions of the theory of random normed spaces as in [17–21]. Throughout this paper, the space of all probability distribution functions is denoted by

$$\begin{aligned} \Delta^+ = \{F : \mathbb{R} \cup \{-\infty, +\infty\} \rightarrow [0, 1] : F \text{ is left-continuous} \\ \text{and nondecreasing on } \mathbb{R} \text{ and } F(0) = 0, F(+\infty) = 1\}, \end{aligned} \quad (1.3)$$

and the subset $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : l^-F(+\infty) = 1\}$, where $l^-f(x)$ denotes the left limit of the function f at the point x . The space Δ^+ is partially ordered by the usual point-wise ordering of functions, that is, $F \leq G$ if and only if $F(t) \leq G(t)$ for all $t \in \mathbb{R}$. The maximal element for Δ^+ in this order is the distribution function given by

$$\varepsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0, \\ 1, & \text{if } t > 0. \end{cases} \quad (1.4)$$

Definition 1.1 (see [17]). A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous triangular norm (briefly, a t -norm) if T satisfies the following conditions:

- (a) T is commutative and associative;
- (b) T is continuous;
- (c) $T(a, 1) = a$ for all $a \in [0, 1]$;
- (d) $T(a, b) \leq T(c, d)$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0, 1]$.

Three typical examples of continuous t -norms are $T(a, b) = ab$, $T(a, b) = \max(a + b - 1, 0)$, and $T(a, b) = \min(a, b)$.

Definition 1.2. A random normed space (briefly, RN-space) is a triple (X, μ, T) , where X is a vector space, T is a continuous t -norm, and μ is a mapping from X into D^+ such that

the following conditions hold:

- (PN1) $\mu_x(t) = \varepsilon_0(t)$ for all $t > 0$ and only if $x = 0$;
- (PN2) $\mu_{\alpha x}(t) = \mu_x(t/|\alpha|)$ for all x in X , $\alpha \neq 0$ and all $t \geq 0$;
- (PN3) $\mu_{x+y}(t+s) \geq T(\mu_x(t), \mu_y(s))$ for all $x, y \in X$ and all $t, s \geq 0$.

Definition 1.3. Let (X, μ, T) be an RN-space.

- (1) A sequence $\{x_n\}$ in X is said to be *convergent* to x in X if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n-x}(t) > 1 - \varepsilon$ whenever $n \geq N$.
- (2) A sequence $\{x_n\}$ in X is called a *Cauchy sequence* if, for every $t > 0$ and $\varepsilon > 0$, there exists a positive integer N such that $\mu_{x_n-x_m}(t) > 1 - \varepsilon$ whenever $n \geq m \geq N$.
- (3) An RN-space (X, μ, T) is said to be *complete* if and only if every Cauchy sequence in X is convergent to a point in X .

Theorem 1.4 (see [20]). *If (X, μ, T) is an RN-space and $\{x_n\}$ is a sequence such that $x_n \rightarrow x$, then $\lim_{n \rightarrow \infty} \mu_{x_n}(t) = \mu_x(t)$.*

Lemma 1.5. *Let (X, μ, \min) be an RN-space and define $E_{\lambda, \mu} : X \rightarrow \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda, \mu}(x) = \inf\{t > 0 : \mu_x(t) > 1 - \lambda\}, \quad \forall \lambda \in]0, 1[, x \in X. \quad (1.5)$$

Then, one has

$$E_{\lambda, \mu}(x_1 - x_n) \leq E_{\lambda, \mu}(x_1 - x_2) + \cdots + E_{\lambda, \mu}(x_{n-1} - x_n), \quad (1.6)$$

for all $x_1, \dots, x_n \in X$ and the sequence $\{x_n\}$ is convergent to x with respect to random norm μ if and only if $E_{\lambda, \mu}(x_n - x) \rightarrow 0$ as $n \rightarrow \infty$. Also, the sequence $\{x_n\}$ is a Cauchy sequence with respect to random norm μ if and only if it is a Cauchy sequence with $E_{\lambda, \mu}$.

Proof. By the triangular inequality, we have

$$\begin{aligned} & \mu_{x_1-x_n}(E_{\lambda, \mu}(x_1 - x_2) + \cdots + E_{\lambda, \mu}(x_{n-1} - x_n) + (n-1)\delta) \\ & \geq \min(\mu_{x_1-x_2}(E_{\lambda, \mu}(x_1 - x_2) + \delta), \dots, \mu_{x_{n-1}-x_n}(E_{\lambda, \mu}(x_{n-1} - x_n) + \delta)) \\ & > \min(1 - \lambda, \dots, 1 - \lambda) \\ & = 1 - \lambda, \quad \forall \delta > 0, \end{aligned} \quad (1.7)$$

which implies that

$$E_{\lambda, \mu}(x_1 - x_n) \leq E_{\lambda, \mu}(x_1 - x_2) + E_{\lambda, \mu}(x_2 - x_3) + \cdots + E_{\lambda, \mu}(x_{n-1} - x_n) + (n-1)\delta. \quad (1.8)$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\lambda, \mu}(x_1 - x_n) \leq E_{\lambda, \mu}(x_1 - x_2) + E_{\lambda, \mu}(x_2 - x_3) + \cdots + E_{\lambda, \mu}(x_{n-1} - x_n). \quad (1.9)$$

Next, we have $\mu_{x_n-x}(\eta) > 1 - \lambda \Leftrightarrow E_{\lambda,\mu}(x_n - x) < \eta$ for every $\eta > 0$. This completes the proof. \square

In this paper, we establish the stability of the cubic and quadratic functional equations in the setting of random normed spaces.

2. On the stability of cubic mappings in RN-spaces

Theorem 2.1. *Let X be a linear space, (Z, μ', \min) an RN-space, and $\varphi : X \times X \rightarrow Z$ a function such that for some $0 < \alpha < 8$,*

$$\mu'_{\varphi(2x,0)}(t) \geq \mu'_{\alpha\varphi(x,0)}(t), \quad \forall x \in X, t > 0, \quad (2.1)$$

$f(0) = 0$ and $\lim_{n \rightarrow \infty} \mu'_{\varphi(2^n x, 2^n y)}(8^n t) = 1$ for all $x, y \in X$ and all $t > 0$. Let (Y, μ, \min) be a complete RN-space. If $f : X \rightarrow Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \mu'_{\varphi(x,y)}(t), \quad \forall x, y \in X, t > 0, \quad (2.2)$$

then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\varphi(x,0)}(2(8-\alpha)t). \quad (2.3)$$

Proof. From (2.2), it follows that

$$\begin{aligned} & E_{\lambda,\mu}(f(2x+y) + f(2x-y) - 2f(x+y) - 2f(x-y) - 12f(x)) \\ &= \inf \{t > 0 : \mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) > 1 - \lambda\} \\ &\leq \inf \{t > 0 : \mu'_{\varphi(x,y)}(t) > 1 - \lambda\} \\ &= E_{\lambda,\mu'}(\varphi(x, y)), \quad \forall x, y \in X, \lambda \in (0, 1). \end{aligned} \quad (2.4)$$

Putting $y = 0$ in (2.4), we get

$$E_{\lambda,\mu}\left(\frac{f(2x)}{8} - f(x)\right) \leq \frac{1}{16} E_{\lambda,\mu'}(\varphi(x, 0)), \quad \forall x \in X. \quad (2.5)$$

Replacing x by $2^n x$ in (2.5) and using (2.1), we obtain

$$\begin{aligned} E_{\lambda,\mu}\left(\frac{f(2^{n+1}x)}{8^{n+1}} - \frac{f(2^n x)}{8^n}\right) &\leq \frac{1}{16 \times 8^n} E_{\lambda,\mu'}(\varphi(2^n x, 0)) \\ &\leq \frac{\alpha^n}{16 \times 8^n} E_{\lambda,\mu'}(\varphi(x, 0)). \end{aligned} \quad (2.6)$$

It follows from $(f(2^n x)/8^n) - f(x) = \sum_{k=0}^{n-1} ((f(2^{k+1} x)/8^{k+1}) - (f(2^k x)/8^k))$ and (2.6) that

$$\begin{aligned}
E_{\lambda, \mu} \left(\frac{f(2^n x)}{8^n} - f(x) \right) &= E_{\lambda, \mu} \left(\sum_{k=0}^{n-1} \left(\frac{f(2^{k+1} x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \right) \right) \\
&\leq \sum_{k=0}^{n-1} E_{\lambda, \mu} \left(\frac{f(2^{k+1} x)}{8^{k+1}} - \frac{f(2^k x)}{8^k} \right) \\
&\leq \sum_{k=0}^{n-1} \frac{1}{16 \times 8^k} E_{\lambda, \mu'}(\varphi(2^k x, 0)) \\
&\leq \sum_{k=0}^{n-1} \frac{\alpha^k}{16 \times 8^k} E_{\lambda, \mu'}(\varphi(x, 0)).
\end{aligned} \tag{2.7}$$

Replacing x with $2^m x$ in (2.7), we observe that

$$\begin{aligned}
E_{\lambda, \mu} \left(\frac{f(2^{n+m} x)}{8^{n+m}} - \frac{f(2^m x)}{8^m} \right) &\leq \sum_{k=0}^{n-1} \frac{\alpha^k}{16 \times 8^{k+m}} E_{\lambda, \mu'}(\varphi(2^m x, 0)) \\
&\leq \sum_{k=0}^{n-1} \frac{\alpha^{k+m}}{16 \times 8^{k+m}} E_{\lambda, \mu'}(\varphi(x, 0)) \\
&\leq \sum_{k=m}^{m+n-1} \frac{\alpha^k}{16 \times 8^k} E_{\lambda, \mu'}(\varphi(x, 0)) \\
&= \frac{E_{\lambda, \mu'}(\varphi(x, 0))}{16} \sum_{k=m}^{m+n-1} \left(\frac{\alpha}{8} \right)^k.
\end{aligned} \tag{2.8}$$

Then $\{f(2^n x)/8^n\}$ is a Cauchy sequence in (Y, μ, \min) . Since (Y, μ, \min) is a complete RN-space, this sequence converges to some point $C(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (2.8). Then we obtain

$$E_{\lambda, \mu} \left(\frac{f(2^n x)}{8^n} - f(x) \right) \leq \frac{E_{\lambda, \mu'}(\varphi(x, 0))}{16} \sum_{k=0}^{n-1} \left(\frac{\alpha}{8} \right)^k, \tag{2.9}$$

and so

$$\begin{aligned}
E_{\lambda, \mu}(C(x) - f(x)) &\leq E_{\lambda, \mu} \left(C(x) - \frac{f(2^n x)}{8^n} \right) + E_{\lambda, \mu} \left(\frac{f(2^n x)}{8^n} - f(x) \right) \\
&\leq E_{\lambda, \mu} \left(C(x) - \frac{f(2^n x)}{8^n} \right) + \frac{E_{\lambda, \mu'}(\varphi(x, 0))}{16} \sum_{k=0}^{n-1} \left(\frac{\alpha}{8} \right)^k.
\end{aligned} \tag{2.10}$$

Taking the limit as $n \rightarrow \infty$ and using (2.10), we get

$$E_{\lambda, \mu}(C(x) - f(x)) \leq \frac{E_{\lambda, \mu'}(\varphi(x, 0))}{16 - 2\alpha}, \quad (2.11)$$

that is,

$$\inf\{t > 0 : \mu_{C(x)-f(x)}(t) > 1 - \lambda\} \leq \inf\{t > 0 : \mu'_{\varphi(x,0)}(2t(8 - \alpha)) > 1 - \lambda\}. \quad (2.12)$$

Then, we have

$$\mu_{C(x)-f(x)}(t) \geq \mu'_{\varphi(x,0)}(2t(8 - \alpha)). \quad (2.13)$$

Replacing x and y by $2^n x$ and $2^n y$ in (2.2), respectively, we get

$$\begin{aligned} & \mu_{f(2^n(2x+y))/8^n + f(2^n(2x-y))/8^n - 2f(2^n(x+y))/8^n - 2f(2^n(x-y))/8^n - 12f(2^n(x))/8^n}(t) \\ & \geq \mu'_{\varphi(2^n x, 2^n y)}(8^n t), \quad \forall x, y \in X, t > 0. \end{aligned} \quad (2.14)$$

Since $\lim_{n \rightarrow \infty} \mu'_{\varphi(2^n x, 2^n y)}(8^n t) = 1$, we conclude that C fulfills (1.1).

To prove the uniqueness of the cubic mapping C , assume that there exists a cubic mapping $D : X \rightarrow Y$ which satisfies (2.3). Fix $x \in X$. Clearly, $C(2^n x) = 8^n C(x)$ and $D(2^n x) = 8^n D(x)$ for all $n \in \mathbb{N}$. It follows from (2.3) that

$$\begin{aligned} \mu_{C(x)-D(x)}(t) &= \lim_{n \rightarrow \infty} \mu_{(C(2^n x)/8^n)-(D(2^n x)/8^n)}(t), \\ \mu_{(C(2^n x)/8^n)-(D(2^n x)/8^n)}(t) &\geq \min \left\{ \mu_{(C(2^n x)/8^n)-(f(2^n x)/8^n)}\left(\frac{t}{2}\right), \mu_{(D(2^n x)/8^n)-(f(2^n x)/8^n)}\left(\frac{t}{2}\right) \right\} \\ &\geq \mu'_{\varphi(2^n x, 0)}(8^n(8 - \alpha)t) \\ &\geq \mu'_{\varphi(x, 0)}\left(\frac{8^n(8 - \alpha)t}{\alpha^n}\right). \end{aligned} \quad (2.15)$$

Since $\lim_{n \rightarrow \infty} (8^n(8 - \alpha)t/\alpha^n) = \infty$, we get $\lim_{n \rightarrow \infty} \mu'_{\varphi(x, 0)}((8^n(8 - \alpha)t)/\alpha^n) = 1$. Therefore, it follows that $\mu_{C(x)-D(x)}(t) = 1$ for all $t > 0$ and so $C(x) = D(x)$. This completes the proof. \square

Corollary 2.2. *Let X be a linear space, (Z, μ', \min) an RN-space, and (Y, μ, \min) a complete RN-space. Let p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that*

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \mu'_{(\|x\|^p + \|y\|^q)z_0}(t), \quad \forall x, y \in X, t > 0, \quad (2.16)$$

$f(0) = 0$ and $p, q < 3$, then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\|x\|^p z_0}(2(8 - 2^p)t), \quad \forall x \in X, t > 0. \quad (2.17)$$

Proof. Let $\varphi : X \times X \rightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then the proof follows from Theorem 2.1 by $\alpha = 2^p$. \square

Corollary 2.3. *Let X be a linear space, (Z, μ', \min) an RN-space, and (Y, μ, \min) a complete RN-space. Let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that*

$$\mu_{f(2x+y)+f(2x-y)-2f(x+y)-2f(x-y)-12f(x)}(t) \geq \mu'_{\varepsilon z_0}(t), \quad \forall x, y \in X, t > 0, \quad (2.18)$$

and $f(0) = 0$, then there exists a unique cubic mapping $C : X \rightarrow Y$ such that

$$\mu_{f(x)-C(x)}(t) \geq \mu'_{\varepsilon z_0}(14t), \quad \forall x \in X, t > 0. \quad (2.19)$$

Proof. Let $\varphi : X \times X \rightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then, the proof follows from Theorem 2.1 by $\alpha = 1$. \square

3. On the stability of quadratic mappings in RN-spaces

Theorem 3.1. *Let X be a linear space, (Z, μ', \min) an RN-space, and $\varphi : X \times X \rightarrow Z$ a function such that for some $0 < \alpha < 16$,*

$$\mu'_{\varphi(2x,0)}(t) \geq \mu'_{\alpha\varphi(x,0)}(t), \quad \forall x \in X, t > 0, \quad (3.1)$$

$f(0) = 0$ and $\lim_{n \rightarrow \infty} \mu'_{\varphi(2^n x, 2^n y)}(16^n t) = 1$ for all $x, y \in X$ and all $t > 0$. Let (Y, μ, \min) be a complete RN-space. If $f : X \rightarrow Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) \geq \mu'_{\varphi(x,y)}(t), \quad \forall x, y \in X, t > 0, \quad (3.2)$$

then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\varphi(x,0)}(2(16-\alpha)t). \quad (3.3)$$

Proof. From (3.2), it follows that

$$\begin{aligned} & E_{\lambda, \mu}(f(2x+y) + f(2x-y) - 4f(x+y) - 4f(x-y) - 24f(x) + 6f(y)) \\ &= \inf \{ t > 0 : \mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) > 1 - \lambda \} \\ &\leq \inf \{ t > 0 : \mu'_{\varphi(x,y)}(t) > 1 - \lambda \} \\ &= E_{\lambda, \mu'}(\varphi(x, y)), \quad \forall x, y \in X, \lambda \in (0, 1). \end{aligned} \quad (3.4)$$

Putting $y = 0$ in (3.4), we get

$$E_{\lambda, \mu} \left(\frac{f(2x)}{16} - f(x) \right) \leq \frac{1}{32} E_{\lambda, \mu'}(\varphi(x, 0)), \quad \forall x \in X. \quad (3.5)$$

Replacing x by $2^n x$ in (3.5) and using (3.1), we obtain

$$\begin{aligned} E_{\lambda,\mu}\left(\frac{f(2^{n+1}x)}{16^{n+1}} - \frac{f(2^n x)}{16^n}\right) &\leq \frac{1}{32 \times 16^n} E_{\lambda,\mu'}(\varphi(2^n x, 0)) \\ &\leq \frac{\alpha^n}{32 \times 16^n} E_{\lambda,\mu'}(\varphi(x, 0)). \end{aligned} \quad (3.6)$$

It follows from $(f(2^n x)/16^n) - f(x) = \sum_{k=0}^{n-1} ((f(2^{k+1}x)/16^{k+1}) - (f(2^k x)/16^k))$ and (3.6) that

$$\begin{aligned} E_{\lambda,\mu}\left(\frac{f(2^n x)}{16^n} - f(x)\right) &= E_{\lambda,\mu}\left(\sum_{k=0}^{n-1} \left(\frac{f(2^{k+1}x)}{16^{k+1}} - \frac{f(2^k x)}{16^k}\right)\right) \\ &\leq \sum_{k=0}^{n-1} E_{\lambda,\mu}\left(\frac{f(2^{k+1}x)}{16^{k+1}} - \frac{f(2^k x)}{16^k}\right) \\ &\leq \sum_{k=0}^{n-1} \frac{1}{32 \times 16^k} E_{\lambda,\mu'}(\varphi(2^k x, 0)) \\ &\leq \sum_{k=0}^{n-1} \frac{\alpha^k}{32 \times 16^k} E_{\lambda,\mu'}(\varphi(x, 0)). \end{aligned} \quad (3.7)$$

Replacing x with $2^m x$ in (3.7), we observe that

$$\begin{aligned} E_{\lambda,\mu}\left(\frac{f(2^{n+m}x)}{16^{n+m}} - \frac{f(2^m x)}{16^m}\right) &\leq \sum_{k=0}^{n-1} \frac{\alpha^k}{32 \times 16^{k+m}} E_{\lambda,\mu'}(\varphi(2^m x, 0)) \\ &\leq \sum_{k=0}^{n-1} \frac{\alpha^{k+m}}{32 \times 16^{k+m}} E_{\lambda,\mu'}(\varphi(x, 0)) \\ &\leq \sum_{k=m}^{m+n-1} \frac{\alpha^k}{32 \times 16^k} E_{\lambda,\mu'}(\varphi(x, 0)) \\ &= \frac{E_{\lambda,\mu'}(\varphi(x, 0))}{32} \sum_{k=m}^{m+n-1} \left(\frac{\alpha}{16}\right)^k. \end{aligned} \quad (3.8)$$

Then $\{f(2^n x)/16^n\}$ is a Cauchy sequence in (Y, μ, \min) . Since (Y, μ, \min) is a complete RN-space, this sequence converges to some point $Q(x) \in Y$. Fix $x \in X$ and put $m = 0$ in (3.8). Then we obtain

$$E_{\lambda,\mu}\left(\frac{f(2^n x)}{16^n} - f(x)\right) \leq \frac{E_{\lambda,\mu'}(\varphi(x, 0))}{32} \sum_{k=0}^{n-1} \left(\frac{\alpha}{16}\right)^k, \quad (3.9)$$

and so

$$\begin{aligned} E_{\lambda,\mu}(Q(x) - f(x)) &\leq E_{\lambda,\mu}\left(Q(x) - \frac{f(2^n x)}{16^n}\right) + E_{\lambda,\mu}\left(\frac{f(2^n x)}{16^n} - f(x)\right) \\ &\leq E_{\lambda,\mu}\left(Q(x) - \frac{f(2^n x)}{16^n}\right) + \frac{E_{\lambda,\mu'}(\varphi(x, 0))}{32} \sum_{k=0}^{n-1} \left(\frac{\alpha}{16}\right)^k. \end{aligned} \quad (3.10)$$

Taking the limit as $n \rightarrow \infty$ and using (3.10), we get

$$E_{\lambda,\mu}(Q(x) - f(x)) \leq \frac{E_{\lambda,\mu'}(\varphi(x, 0))}{32 - 2\alpha}, \quad (3.11)$$

that is,

$$\inf\{t > 0 : \mu_{Q(x)-f(x)}(t) > 1 - \lambda\} \leq \inf\{t > 0 : \mu'_{\varphi(x,0)}(2t(16 - \alpha)) > 1 - \lambda\}. \quad (3.12)$$

Then, we have

$$\mu_{Q(x)-f(x)}(t) \geq \mu'_{\varphi(x,0)}(2t(16 - \alpha)). \quad (3.13)$$

Replacing x and y by $2^n x$ and $2^n y$ in (3.2), respectively, we get

$$\begin{aligned} &\mu_{f(2^n(2x+y))/16^n + f(2^n(2x-y))/16^n - 4f(2^n(x+y))/16^n - 4f(2^n(x-y))/16^n - 24f(2^n(x))/16^n + 6f(2^n(y))/16^n}(t) \\ &\geq \mu'_{\varphi(2^n x, 2^n y)}(16^n t), \quad \forall x, y \in X, t > 0. \end{aligned} \quad (3.14)$$

Since $\lim_{n \rightarrow \infty} \mu'_{\varphi(2^n x, 2^n y)}(16^n t) = 1$, we conclude that Q fulfills (1.2).

To prove the uniqueness of the quadratic mapping Q , assume that there exists a quadratic mapping $D : X \rightarrow Y$ which satisfies (3.3). Fix $x \in X$. Clearly, $Q(2^n x) = 16^n Q(x)$ and $D(2^n x) = 16^n D(x)$ for all $n \in \mathbb{N}$. It follows from (3.3) that

$$\begin{aligned} \mu_{Q(x)-D(x)}(t) &= \lim_{n \rightarrow \infty} \mu_{(Q(2^n x)/16^n) - (D(2^n x)/16^n)}(t), \\ \mu_{(Q(2^n x)/16^n) - (D(2^n x)/16^n)}(t) &\geq \min \left\{ \mu_{(Q(2^n x)/16^n) - (f(2^n x)/16^n)}\left(\frac{t}{2}\right), \mu_{(D(2^n x)/8^n) - (f(2^n x)/8^n)}\left(\frac{t}{2}\right) \right\} \\ &\geq \mu'_{\varphi(2^n x, 0)}(16^n (16 - \alpha)t) \\ &\geq \mu'_{\varphi(x, 0)}\left(\frac{16^n (16 - \alpha)t}{\alpha^n}\right). \end{aligned} \quad (3.15)$$

Since $\lim_{n \rightarrow \infty} (16^n (16 - \alpha)t / \alpha^n) = \infty$, we get $\lim_{n \rightarrow \infty} \mu'_{\varphi(x, 0)}(16^n (16 - \alpha)t / \alpha^n) = 1$. Therefore, it follows that $\mu_{Q(x)-D(x)}(t) = 1$ for all $t > 0$ and so $Q(x) = D(x)$. This completes the proof. \square

Corollary 3.2. Let X be a linear space, (Z, μ', \min) an RN-space, and (Y, μ, \min) a complete RN-space. Let p, q be nonnegative real numbers and let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) \geq \mu'_{(\|x\|^p + \|y\|^q)z_0}(t), \quad \forall x, y \in X, t > 0, \quad (3.16)$$

$f(0) = 0$ and $p, q < 4$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\|x\|^p z_0}(2(16 - 2^p)t), \quad \forall x \in X, t > 0. \quad (3.17)$$

Proof. Let $\varphi : X \times X \rightarrow Z$ be defined by $\varphi(x, y) = (\|x\|^p + \|y\|^q)z_0$. Then, the proof follows from Theorem 3.1 by $\alpha = 2^p$. \square

Corollary 3.3. Let X be a linear space, (Z, μ', \min) an RN-space, and (Y, μ, \min) a complete RN-space. Let $z_0 \in Z$. If $f : X \rightarrow Y$ is a mapping such that

$$\mu_{f(2x+y)+f(2x-y)-4f(x+y)-4f(x-y)-24f(x)+6f(y)}(t) \geq \mu'_{\varepsilon z_0}(t), \quad \forall x, y \in X, t > 0, \quad (3.18)$$

and $f(0) = 0$, then there exists a unique quadratic mapping $Q : X \rightarrow Y$ such that

$$\mu_{f(x)-Q(x)}(t) \geq \mu'_{\varepsilon z_0}(30t), \quad \forall x \in X, t > 0. \quad (3.19)$$

Proof. Let $\varphi : X \times X \rightarrow Z$ be defined by $\varphi(x, y) = \varepsilon z_0$. Then, the proof follows from Theorem 3.1 by $\alpha = 1$. \square

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