Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2008, Article ID 305623, 12 pages
doi:10.1155/2008/305623

Research Article

On Logarithmic Convexity for Power Sums and Related Results II

J. Pečarić[1, 2] and Atiq ur Rehman[1]

[1] *Abdus Salam School of Mathematical Sciences, GC University, Lahore 54600, Pakistan*
[2] *Faculty of Textile Technology, University of Zagreb, Pierottijeva 6, 10000 Zagreb, Croatia*

Correspondence should be addressed to Atiq ur Rehman, mathcity@gmail.com

Received 14 October 2008; Accepted 4 December 2008

Recommended by Wing-Sum Cheung

In the paper "On logarithmic convexity for power sums and related results" (2008), we introduced means by using power sums and increasing function. In this paper, we will define new means of convex type in connection to power sums. Also we give integral analogs of new means.

1. Introduction and preliminaries

Let $\mathbf{x}$ be positive n-tuples. The well-known inequality for power sums of order s and r, for $s > r > 0$ (see [1, page 164]), states that

$$\left(\sum_{i=1}^{n} x_i^s\right)^{1/s} < \left(\sum_{i=1}^{n} x_i^r\right)^{1/r}. \tag{1.1}$$

Moreover, if $\mathbf{p} = (p_1, \dots, p_n)$ is a positive n-tuples such that $p_i \geq 1$ $(i = 1, \dots, n)$, then for $s > r > 0$ (see [1, page 165]), we have

$$\left(\sum_{i=1}^{n} p_i x_i^s\right)^{1/s} < \left(\sum_{i=1}^{n} p_i x_i^r\right)^{1/r}. \tag{1.2}$$

In [2], we defined the following function:

$$\Delta_t = \Delta_t(\mathbf{x}; \mathbf{p}) = \begin{cases} \dfrac{1}{t-1}\left(\left(\displaystyle\sum_{i=1}^{n} p_i x_i\right)^t - \displaystyle\sum_{i=1}^{n} p_i x_i^t\right), & t \neq 1, \\ \displaystyle\sum_{i=1}^{n} p_i x_i \log \sum_{i=1}^{n} p_i x_i - \sum_{i=1}^{n} p_i x_i \log x_i, & t = 1. \end{cases} \tag{1.3}$$

We introduced the Cauchy means involving power sums. Namely, the following results were obtained in [2].

For $r < s < t$, where $r, s, t \in \mathbb{R}^+$, we have

$$(\Delta_s)^{t-r} \leq (\Delta_r)^{t-s}(\Delta_t)^{s-r}, \tag{1.4}$$

such that, $x_i \in (0, a]$ $(i = 1, \ldots, n)$ and

$$\sum_{i=1}^{n} p_i x_i \geq x_j, \quad \text{for } j = 1, \ldots, n, \qquad \sum_{i=1}^{n} p_i x_i \in [0, a]. \tag{1.5}$$

We defined the following means.

Definition 1.1. Let $\mathbf{x}$ and $\mathbf{p}$ be two nonnegative n-tuples $(n \geq 2)$ such that $p_i \geq 1$ $(i = 1, \ldots, n)$. Then for $t, r, s \in \mathbb{R}^+$,

$$\begin{aligned}
A^s_{t,r}(\mathbf{x};\mathbf{p}) &= \left\{ \frac{r-s}{t-s} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{t/s} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r} \right\}^{1/(t-r)}, \quad t \neq r,\ r \neq s,\ t \neq s, \\
A^s_{s,r}(\mathbf{x};\mathbf{p}) = A^s_{r,s}(\mathbf{x};\mathbf{p}) &= \left\{ \frac{r-s}{s} \frac{\left(\sum_{i=1}^n p_i x_i^s\right) \log \sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r} \right\}^{1/(s-r)}, \quad s \neq r, \\
A^s_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(\frac{1}{s-r} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} \log \sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r\right\}} \right), \quad s \neq r, \\
A^s_{s,s}(\mathbf{x};\mathbf{p}) &= \exp\left(\frac{\left(\sum_{i=1}^n p_i x_i^s\right)\left(\log \sum_{i=1}^n p_i x_i^s\right)^2 - s^2 \sum_{i=1}^n p_i x_i^s \left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right) \log\left(\sum_{i=1}^n p_i x_i^s\right) - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}} \right).
\end{aligned} \tag{1.6}$$

In this paper, we introduce new Cauchy means of convex type in connection with Power sums. For means, we shall use the following result [1, page 154].

Theorem 1.2. *Let* $\mathbf{x}$ *and* $\mathbf{p}$ *be two nonnegative n-tuples such that condition* (1.5) *is valid. If* f *is a convex function on* $[0, a]$, *then*

$$f\left(\sum_{i=1}^n p_i x_i\right) \geq \sum_{i=1}^n p_i f(x_i) + \left(1 - \sum_{i=1}^n p_i\right) f(0). \tag{1.7}$$

Remark 1.3. In Theorem 1.2, if f is strictly convex, then (1.7) is strict unless $x_1 = \cdots = x_n$ and $\sum_{i=1}^n p_i = 1$.

2. Discrete result

Lemma 2.1. *Let*

$$\varphi_t(x) = \begin{cases} \dfrac{x^t}{t(t-1)}, & t \neq 1, \\ x \log x, & t = 1, \end{cases} \tag{2.1}$$

where $t \in \mathbb{R}^+$. Then $\varphi_t(x)$ is strictly convex for $x > 0$.

Here, we use the notation $0 \log 0 := 0$.

Proof. Since $\varphi_t''(x) = x^{t-2} > 0$ for $x > 0$, therefore $\varphi_t(x)$ is strictly convex for $x > 0$. □

Lemma 2.2 (see [3]). *A positive function f is* log *convex in Jensen sense on an open interval I, that is, for each $s, t \in I$*

$$f(s) f(t) \geq f^2\left(\frac{s+t}{2}\right), \tag{2.2}$$

if and only if the relation

$$u^2 f(s) + 2uw f\left(\frac{s+t}{2}\right) + w^2 f(t) \geq 0, \tag{2.3}$$

holds for each real u, w and $s, t \in I$.

The following lemma is equivalent to definition of convex function [1, page 2].

Lemma 2.3. *If f is continuous and convex for all x_1, x_2, x_3 of an open interval I for which $x_1 < x_2 < x_3$, then*

$$(x_3 - x_2) f(x_1) + (x_1 - x_3) f(x_2) + (x_2 - x_1) f(x_3) \geq 0. \tag{2.4}$$

Lemma 2.4. *Let f be* log-convex *function and if, $x_1 \leq y_1$, $x_2 \leq y_2$, $x_1 \neq x_2$, $y_1 \neq y_2$, then the following inequality is valid:*

$$\left(\frac{f(x_2)}{f(x_1)}\right)^{1/(x_2 - x_1)} \leq \left(\frac{f(y_2)}{f(y_1)}\right)^{1/(y_2 - y_1)}. \tag{2.5}$$

By using the above lemmas and Theorem 1.2, as in [2], we can prove the following results.

Theorem 2.5. *Let $\mathbf{x}$ and $\mathbf{p}$ be two positive n-tuples and let*

$$\overline{\Delta}_t = \overline{\Delta}_t(\mathbf{x}; \mathbf{p}) = \frac{\Delta_t}{t}, \tag{2.6}$$

such that condition (1.5) *is satisfied and all* x_i*'s are not equal. Then* $\overline{\Delta}_t$ *is* log-convex*. Also for* $r < s < t$ *where* $r, s, t \in \mathbb{R}^+$*, we have*

$$\left(\overline{\Delta}_s\right)^{t-r} \le \left(\overline{\Delta}_r\right)^{t-s} \left(\overline{\Delta}_t\right)^{s-r}. \tag{2.7}$$

Moreover, we can use (2.7) to obtain new means of Cauchy type involving power sums.

Let us introduce the following means.

Definition 2.6. Let $\mathbf{x}$ and $\mathbf{p}$ be two nonnegative n-tuples such that $p_i \ge 1$ $(i = 1, \dots, n)$, then for $t, r, s \in \mathbb{R}^+$,

$$\begin{aligned}
B^s_{t,r}(\mathbf{x};\mathbf{p}) &= \left\{ \frac{r(r-s)}{t(t-s)} \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{t/s} - \sum_{i=1}^n p_i x_i^t}{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r} \right\}^{1/(t-r)}, \quad t \neq r,\ r \neq s,\ t \neq s, \\
B^s_{s,r}(\mathbf{x};\mathbf{p}) = B^s_{r,s}(\mathbf{x};\mathbf{p}) &= \left\{ \frac{r(r-s)}{s^2} \frac{\left(\sum_{i=1}^n p_i x_i^s\right) \log \sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i}{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r} \right\}^{1/(s-r)}, \quad s \neq r, \\
B^s_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{2r-s}{r(r-s)} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} \log \sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^r \log x_i}{s\left\{\left(\sum_{i=1}^n p_i x_i^s\right)^{r/s} - \sum_{i=1}^n p_i x_i^r\right\}} \right), \quad s \neq r, \\
B^s_{s,s}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{s} + \frac{\left(\sum_{i=1}^n p_i x_i^s\right) \left(\log \sum_{i=1}^n p_i x_i^s\right)^2 - s^2\sum_{i=1}^n p_i x_i^s \left(\log x_i\right)^2}{2s\left\{\left(\sum_{i=1}^n p_i x_i^s\right) \log \sum_{i=1}^n p_i x_i^s - s\sum_{i=1}^n p_i x_i^s \log x_i\right\}} \right).
\end{aligned} \tag{2.8}$$

Remark 2.7. Let us note that $B^s_{s,r}(\mathbf{x};\mathbf{p}) = B^s_{r,s}(\mathbf{x};\mathbf{p}) = \lim_{t\to s} B^s_{t,r}(\mathbf{x};\mathbf{p}) = \lim_{t\to s} B^s_{r,t}(\mathbf{x};\mathbf{p})$, $B^s_{r,r}(\mathbf{x};\mathbf{p}) = \lim_{t\to r} B^s_{t,r}(\mathbf{x};\mathbf{p})$ and $B^s_{s,s}(\mathbf{x};\mathbf{p}) = \lim_{r\to s} B^s_{r,r}(\mathbf{x};\mathbf{p})$.

Theorem 2.8. *Let*

$$\Theta^s_t = \begin{cases} \dfrac{1}{t(t-s)}\left\{\left(\displaystyle\sum_{i=1}^n p_i x_i^s\right)^{t/s} - \displaystyle\sum_{i=1}^n p_i x_i^t\right\}, & t \neq s, \\ \dfrac{1}{s^2}\left\{\left(\displaystyle\sum_{i=1}^n p_i x_i^s\right) \log\left(\displaystyle\sum_{i=1}^n p_i x_i^s\right) - s\displaystyle\sum_{i=1}^n p_i x_i^s \log x_i\right\}, & t = s. \end{cases} \tag{2.9}$$

then for $t, r, u \in \mathbb{R}^+$ *and* $t < r < u$*, we have*

$$\left(\Theta^s_r\right)^{u-t} \le \left(\Theta^s_t\right)^{u-r} \left(\Theta^s_u\right)^{r-t}. \tag{2.10}$$

Theorem 2.9. *Let* $r, t, u, v \in \mathbb{R}^+$*, such that* $t \le v$*,* $r \le u$*. Then one has*

$$B^s_{t,r}(\mathbf{x};\mathbf{p}) \le B^s_{v,u}(\mathbf{x};\mathbf{p}). \tag{2.11}$$

Remark 2.10. From (2.7), we have

$$\left(\frac{\Delta_s}{s}\right)^{t-r} \le \left(\frac{\Delta_r}{r}\right)^{t-s}\left(\frac{\Delta_t}{t}\right)^{s-r} \Longrightarrow (\Delta_s)^{t-r} \le \frac{s^{t-r}}{r^{t-s}t^{s-r}}(\Delta_r)^{t-s}(\Delta_t)^{s-r}. \tag{2.12}$$

Since $\log x$ is concave, therefore for $r < s < t$, we have

$$(t-s)\log r + (r-t)\log s + (s-r)\log t < 0 \Longrightarrow \frac{s^{t-r}}{r^{t-s}t^{s-r}} > 1. \tag{2.13}$$

This implies that (1.4), which we derived in [2], is better than (2.7).

Also note that

$$\begin{aligned} B^s_{t,r}(\mathbf{x};\mathbf{p}) &= \left(\frac{r}{t}\right)^{1/(t-r)} A^s_{t,r}(\mathbf{x};\mathbf{p}), \\ B^s_{r,s}(\mathbf{x};\mathbf{p}) = B^s_{s,r}(\mathbf{x};\mathbf{p}) &= \left(\frac{r}{s}\right)^{1/(s-r)} A^s_{s,r}(\mathbf{x};\mathbf{p}) = \left(\frac{r}{s}\right)^{1/(s-r)} A^s_{r,s}(\mathbf{x};\mathbf{p}), \\ B^s_{r,r}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{r}\right) A^s_{r,r}(\mathbf{x};\mathbf{p}), \\ B^s_{s,s}(\mathbf{x};\mathbf{p}) &= \exp\left(-\frac{1}{s}\right) A^s_{s,s}(\mathbf{x};\mathbf{p}). \end{aligned} \tag{2.14}$$

Let us note that there are not integral analogs of results from [2]. Moreover, in Section 3 we will show that previous results have their integral analogs.

3. Integral results

The following theorem is very useful for further result [1, page 159].

Theorem 3.1. *Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation and*

$$G(t) := \int_a^t dg(x), \qquad \overline{G}(t) := \int_t^b dg(x). \tag{3.1}$$

(a) *If*

$$0 \le G(t) \le 1 \quad \text{for } a \le t \le t_0, \qquad 0 \le \overline{G}(t) \le 1 \quad \text{for } t_0 \le t \le b, \tag{3.2}$$

then for every convex function $f : I \to \mathbb{R}$ such that $h(x) \in I$ for all $x \in [a,b]$,

$$\int_a^b f(h(t))dg(t) \ge f\left(\int_a^b h(t)dg(t)\right) + \left(\int_a^b dg(t) - 1\right) f(0). \tag{3.3}$$

(b) *If* $\int_a^b h(t)dg(t) \in I$ *and either there exists an* $s \le t_0$ *such that*

$$G(t) \le 0 \quad \text{for } t < s, \qquad G(t) \ge 1 \quad \text{for } s \le t \le t_0, \qquad \overline{G}(t) \le 0 \quad \text{for } t > t_0, \tag{3.4}$$

or there exists an $s \ge t_0$ *such that*

$$G(t) \le 0 \quad \text{for } t < t_0, \qquad \overline{G}(t) \ge 1 \quad \text{for } t_0 < t < s, \qquad \overline{G}(t) \le 0 \quad \text{for } t \ge s, \tag{3.5}$$

then for every convex function $f : I \to \mathbb{R}$ *such that* $h(x) \in I$ *for all* $x \in [a,b]$, *the reverse of the inequality in* (3.3) *holds.*

To define the new means of Cauchy involving integrals, we define the following function.

Definition 3.2. Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation. Choose g such that function Λ_t is positive valued, where Λ_t is defined as follows:

$$\Lambda_t = \Lambda_t(a,b,h,g) = \int_a^b \varphi_t(h(x))dg(x) - \varphi_t\left(\int_a^b h(x)dg(x)\right). \tag{3.6}$$

Theorem 3.3. *Let* Λ_t*, defined as above, satisfy condition* (3.2)*. Then* Λ_t *is* log*-convex. Also for* $r < s < t$*, where* $r,s,t \in \mathbb{R}^+$*, one has*

$$(\Lambda_s)^{t-r} \le (\Lambda_r)^{t-s}(\Lambda_t)^{s-r}. \tag{3.7}$$

Proof. Let $f(x) = u^2\varphi_s(x) + 2uw\varphi_r(x) + w^2\varphi_t(x)$, where $r = (s+t)/2$ and $u,w \in \mathbb{R}$,

$$f''(x) = u^2x^{s-2} + 2uwx^{r-2} + w^2x^{t-2} = \left(ux^{(s-2)/2} + wx^{(t-2)/2}\right)^2 \ge 0. \tag{3.8}$$

This implies that $f(x)$ is convex.

By Theorem 3.1, we have,

$$\begin{aligned}
&\int_a^b f(h(t))dg(t) - f\left(\int_a^b h(t)dg(t)\right) - \left(\int_a^b dg(t) - 1\right)f(0) \ge 0 \\
&\quad\Longrightarrow u^2\left(\int_a^b \varphi_s(h(x))dg(x) - \varphi_s\left(\int_a^b h(x)dg(x)\right)\right) \\
&\qquad + 2uw\left(\int_a^b \varphi_r(h(x))dg(x) - \varphi_r\left(\int_a^b h(x)dg(x)\right)\right) \\
&\qquad + 2w^2\left(\int_a^b \varphi_t(h(x))dg(x) - \varphi_t\left(\int_a^b h(x)dg(x)\right)\right) \ge 0 \\
&\quad\Longrightarrow u^2\Lambda_s + 2uw\Lambda_r + w^2\Lambda_t \ge 0.
\end{aligned} \tag{3.9}$$

Now, by Lemma 2.2, we have Λ_t is log-convex in Jensen sense.

Since $\lim_{t\to 1}\Lambda_t = \Lambda_1$, this implies that Λ_t is continuous for all $t \in \mathbb{R}^+$, therefore it is a log-convex [1, page 6].

Since Λ_t is log-convex, that is, $\log \Lambda_t$ is convex, therefore by Lemma 2.3 for $r < s < t$ and taking $f = \log \Lambda$, we have

$$(t-s)\log \Lambda_r + (r-t)\log \Lambda_s + (s-r)\log \Lambda_t \geq 0, \tag{3.10}$$

which is equivalent to (3.7). □

Theorem 3.4. *Let* $\widetilde{\Lambda}_t = -\Lambda_t$ *such that condition* (3.4) *or* (3.5) *is satisfied. Then* $\widetilde{\Lambda}_t$ *is* log-*convex. Also for* $r < s < t$*, where* $r, s, t \in \mathbb{R}^+$*, one has*

$$\left(\widetilde{\Lambda}_s\right)^{t-r} \leq \left(\widetilde{\Lambda}_r\right)^{t-s}\left(\widetilde{\Lambda}_t\right)^{s-r}. \tag{3.11}$$

Definition 3.5. Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation. Then for $t, r, s \in \mathbb{R}^+$, one defines

$$\begin{aligned}
&F^s_{t,r}(a,b,h,g)\\
&= \left\{\frac{r(r-s)}{t(t-s)}\frac{\int_a^b h^t(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^{t/s}}{\int_a^b h^r(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^{r/s}}\right\}^{1/(t-r)}, \quad t \neq r,\ r \neq s,\ t \neq s,\\
&F^s_{s,r}(a,b,h,g)\\
&= F^s_{r,s}(a,b,h,g)\\
&= \left\{\frac{r(r-s)}{s^2}\frac{s\int_a^b h^s(x)\log h(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)\log \int_a^b h^s(x)dg(x)}{\int_a^b h^r(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{r/s}}\right\}^{1/(s-r)}, \quad s \neq r,\\
&F^s_{r,r}(a,b,h,g)\\
&= \exp\left(-\frac{2r-s}{r(r-s)} + \frac{s\int_a^b h^r(x)\log h(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{r/s}\log \int_a^b h^s(x)dg(x)}{s\left\{\int_a^b h^r(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{r/s}\right\}}\right), \quad s \neq r,\\
&F^s_{s,s}(a,b,h,g)\\
&= \exp\left(-\frac{1}{s} + \frac{s^2\int_a^b h^s(x)\left(\log h(x)\right)^2 dg(x) - \left(\int_a^b h^s(x)dg(x)\right)\left(\log \int_a^b h^s(x)dg(x)\right)^2}{2s\left\{s\int_a^b h^s(x)\log h(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)\log \int_a^b h^s(x)dg(x)\right\}}\right).
\end{aligned} \tag{3.12}$$

Remark 3.6. Let us note that $F^s_{s,r}(a,b,h,g) = F^s_{r,s}(a,b,h,g) = \lim_{t\to s} F^s_{t,r}(a,b,h,g) = \lim_{t\to s} F^s_{r,t}(a,b,h,g)$, $F^s_{r,r}(a,b,h,g) = \lim_{t\to r} F^s_{t,r}(a,b,h,g)$ and $F^s_{s,s}(a,b,h,g) = \lim_{r\to s} F^s_{r,r}(a,b,h,g)$.

Theorem 3.7. *Let $r, t, u, v \in \mathbb{R}^+$, such that $t \leq v$, $r \leq u$. Then*

$$F^s_{t,r}(a,b,h,g) \leq F^s_{v,u}(a,b,h,g). \tag{3.13}$$

Proof. Let

$$\Lambda_t = \Lambda_t(a,b,h,g) = \begin{cases} \dfrac{1}{t(t-1)}\left(\displaystyle\int_a^b h^t(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^t\right), & t \neq 1, \\ \displaystyle\int_a^b h(x)\,\log\, h(x)dg(x) - \int_a^b h(x)dg(x)\,\log\int_a^b h(x)dg(x), & t = 1. \end{cases} \tag{3.14}$$

Now, taking $x_1 = r$, $x_2 = t$, $y_1 = u$, $y_2 = v$, where $r, t, u, v \neq 1$, and $f(t) = \Lambda_t$ in Lemma 2.4, we have

$$\left(\frac{r(r-1)}{t(t-1)}\frac{\int_a^b h^t(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^t}{\int_a^b h^r(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^r}\right)^{1/(t-r)} \leq \left(\frac{u(u-1)}{v(v-1)}\frac{\int_a^b h^v(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^v}{\int_a^b h^u(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^u}\right)^{1/(v-u)}. \tag{3.15}$$

Since $s > 0$, by substituting $h = h^s$, $t = t/s$, $r = r/s$, $u = u/s$, and $v = v/s$, where $r, t, v, u \neq s$, in above inequality, we get

$$\left(\frac{r(r-s)}{t(t-s)}\frac{\int_a^b h^t(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{t/s}}{\int_a^b h^r(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{r/s}}\right)^{s/(t-r)} \leq \left(\frac{u(u-s)}{v(v-s)}\frac{\int_a^b h^v(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{v/s}}{\int_a^b h^u(x)dg(x) - \left(\int_a^b h^s(x)dg(x)\right)^{u/s}}\right)^{s/(v-u)}. \tag{3.16}$$

By raising power $1/s$, we get an inequality (3.13) for $r, t, v, u \neq s$.

From Remark 3.6 , we get (3.13) is also valid for $r = s$ or $t = s$ or $r = t$ or $t = r = s$. □

Lemma 3.8. *Let $f \in C^2(I)$ such that*

$$m \leq f''(x) \leq M. \tag{3.17}$$

Consider the functions ϕ_1, ϕ_2 defined as

$$\begin{aligned}\phi_1(x) &= \frac{Mx^2}{2} - f(x),\\ \phi_2(x) &= f(x) - \frac{mx^2}{2}.\end{aligned} \tag{3.18}$$

Then $\phi_i(x)$ for $i = 1, 2$ are convex.

Proof. We have that

$$\begin{aligned}\phi_1''(x) &= M - f''(x) \geq 0,\\ \phi_2''(x) &= f''(x) - m \geq 0,\end{aligned} \tag{3.19}$$

that is, ϕ_i for $i = 1, 2$ are convex. □

Theorem 3.9. *Let $t_0 \in [a, b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation, and $f \in C^2(I)$ such that condition (3.2) is satisfied. Then there exists $\xi \in I$ such that*

$$\begin{aligned}&\int_a^b f(h(x))dg(x) - f\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)\\ &\quad = \frac{f''(\xi)}{2}\left\{\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2\right\}.\end{aligned} \tag{3.20}$$

Proof. In Theorem 3.1, setting $f = \phi_1$ and $f = \phi_2$, respectively, as defined in Lemma 3.8, we get the following inequalities:

$$\begin{aligned}&\int_a^b f(h(x))dg(x) - f\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)\\ &\quad \leq \frac{M}{2}\left\{\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2\right\},\end{aligned} \tag{3.21}$$

$$\begin{aligned}&\int_a^b f(h(x))dg(x) - f\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)\\ &\quad \geq \frac{m}{2}\left\{\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2\right\}.\end{aligned} \tag{3.22}$$

Now, by combining both inequalities, we get

$$m \leq \frac{2\left\{\int_a^b f(h(x))dg(x) - f\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)f(0)\right\}}{\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2} \leq M. \tag{3.23}$$

So by condition (3.17), there exists $\xi \in I$ such that

$$\frac{2\{\int_a^b f(h(x))dg(x) - f(\int_a^b h(x)dg(x)) - (\int_a^b dg(x) - 1)f(0)\}}{\int_a^b h^2(x)dg(x) - (\int_a^b h(x)dg(x))^2} = f''(\xi), \tag{3.24}$$

and (3.24) implies (3.20).

Moreover, (3.21) is valid if f'' is bounded from above and again we have (3.20) is valid. Of course (3.20) is obvious if f'' is not bounded from above and below as well. □

Theorem 3.10. *Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation, and $f_1, f_2 \in C^2(I)$ such that condition (3.2) is satisfied. Then there exists $\xi \in I$ such that the following equality is true:*

$$\frac{\int_a^b f_1(h(x))dg(x) - f_1(\int_a^b h(x)dg(x)) - (\int_a^b dg(x) - 1)f_1(0)}{\int_a^b f_2(h(x))dg(x) - f_2(\int_a^b h(x)dg(x)) - (\int_a^b dg(x) - 1)f_2(0)} = \frac{f_1''(\xi)}{f_2''(\xi)}, \tag{3.25}$$

provided that denominators are nonzero.

Proof. Let a function $k \in C^2(I)$ be defined as

$$k = c_1 f_1 - c_2 f_2, \tag{3.26}$$

where c_1 and c_2 are defined as

$$\begin{aligned} c_1 &= \int_a^b f_2(h(x))dg(x) - f_2\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)f_2(0), \\ c_2 &= \int_a^b f_1(h(x))dg(x) - f_1\left(\int_a^b h(x)dg(x)\right) - \left(\int_a^b dg(x) - 1\right)f_1(0). \end{aligned} \tag{3.27}$$

Then, using Theorem 3.9 with $f = k$, we have

$$0 = \left(c_1 f_1''(\xi) - c_2 f_2''(\xi)\right)\left\{\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2\right\}. \tag{3.28}$$

Since

$$\int_a^b h^2(x)dg(x) - \left(\int_a^b h(x)dg(x)\right)^2 \neq 0, \tag{3.29}$$

therefore, (3.28) gives

$$\frac{c_2}{c_1} = \frac{f_1''(\xi)}{f_2''(\xi)}. \tag{3.30}$$

After putting values, we get (3.25). □

Let α be a strictly monotone continuous function, we defined $T_\alpha(h,g)$ as follows (integral version of quasiarithmetic sum [2]):

$$T_\alpha(h,g) = \alpha^{-1}\left(\int_a^b \alpha(h(x))dg(x)\right). \tag{3.31}$$

Theorem 3.11. *Let $\alpha, \beta, \gamma \in C^2[a,b]$ be strictly monotonic continuous functions. Then there exists η in the image of $h(x)$ such that*

$$\frac{\alpha(T_\alpha(h,g)) - \alpha(T_\gamma(h,g)) - \left(\int_a^b dg(x) - 1\right)\alpha\circ\gamma^{-1}(0)}{\beta(T_\beta(h,g)) - \beta(T_\gamma(h,g)) - \left(\int_a^b dg(x) - 1\right)\beta\circ\gamma^{-1}(0)} = \frac{\alpha''(\eta)\gamma'(\eta) - \alpha'(\eta)\gamma''(\eta)}{\beta''(\eta)\gamma'(\eta) - \beta'(\eta)\gamma''(\eta)} \tag{3.32}$$

is valid, provided that all denominators are nonzero.

Proof. If we choose the functions f_1 and f_2 so that $f_1 = \alpha\circ\gamma^{-1}$, $f_2 = \beta\circ\gamma^{-1}$, and $h(x) \to \gamma(h(x))$. Substituting these in (3.25),

$$\begin{aligned}
&\frac{\alpha(T_\alpha(h,g)) - \alpha(T_\gamma(h,g)) - \left(\int_a^b dg(x) - 1\right)\alpha\circ\gamma^{-1}(0)}{\beta(T_\beta(h,g)) - \beta(T_\gamma(h,g)) - \left(\int_a^b dg(x) - 1\right)\beta\circ\gamma^{-1}(0)} \\
&\quad = \frac{\alpha''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \alpha'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))}{\beta''(\gamma^{-1}(\xi))\gamma'(\gamma^{-1}(\xi)) - \beta'(\gamma^{-1}(\xi))\gamma''(\gamma^{-1}(\xi))}.
\end{aligned} \tag{3.33}$$

Then by setting $\gamma^{-1}(\xi) = \eta$, we get (3.32). □

Corollary 3.12. *Let $t_0 \in [a,b]$ be fixed, h be continuous and monotonic with $h(t_0) = 0$, g be a function of bounded variation, and let $t, r, s \in \mathbb{R}^+$. Then*

$$F_{t,r}^s(a,b,h,g) = \eta. \tag{3.34}$$

Proof. If t, r, and s are pairwise distinct, then we put $\alpha(x) = x^t$, $\beta(x) = x^r$ and $\gamma(x) = x^s$ in (3.32) to get (3.34).

For other cases, we can consider limit as in Remark 3.6. □

Acknowledgments

This research was partially funded by Higher Education Commission, Pakistan. The research of the first author was supported by the Croatian Ministry of Science, Education and Sports under the research Grant 117-1170889-0888.

References

[1] J. Pečarić, F. Proschan, and Y. L. Tong, *Convex Functions, Partial Orderings, and Statistical Applications*, vol. 187 of *Mathematics in Science and Engineering*, Academic Press, Boston, Mass, USA, 1992.

[2] J. Pečarić and A. U. Rehman, "On logarithmic convexity for power sums and related results," *Journal of Inequalities and Applications*, vol. 2008, Article ID 389410, 9 pages, 2008.

[3] S. Simic, "On logarithmic convexity for differences of power means," *Journal of Inequalities and Applications*, vol. 2007, Article ID 37359, 8 pages, 2007.