Hindawi Publishing Corporation
Journal of Inequalities and Applications
Volume 2007, Article ID 82138, 10 pages
doi:10.1155/2007/82138

Research Article

Hilbert's Type Linear Operator and Some Extensions of Hilbert's Inequality

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Received 17 April 2007; Accepted 3 October 2007

Recommended by Ram N. Mohapatra

The norm of a Hilbert's type linear operator $T : L^2(0,\infty) \to L^2(0,\infty)$ is given. As applications, a new generalizations of Hilbert integral inequality, and the result of series analogues are given correspondingly.

1. Introduction

At the close of the 19th century a theorem of great elegance and simplicity was discovered by D. Hilbert.

Theorem 1.1 (Hilbert's double series theorem). *The series*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n} \tag{1.1}$$

is convergent whenever $\sum_{n=1}^{\infty} a_n^2$ is convergent.

The Hilbert's inequalities were studied extensively; refinements, generalizations, and numerous variants appeared in the literature (see [1, 2]). Firstly, we will recall some Hilbert's inequalities. If $f(x), g(x) \geq 0$, $0 < \int_0^\infty f^2(x)dx < \infty$ and $0 < \int_0^\infty g^2(x)dx < \infty$, then

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy < \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.2}$$

where the constant factor π is the best possible. Inequality (1.2) is named of Hardy-Hilbert's integral inequality (see [3]). Under the same condition of (1.2), we have the

Hardy-Hilbert's type inequality (see [3], Theorem 319, Theorem 341) similar to (1.2):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}}dx\,dy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{1.3}$$

where the constant factor 4 is also the best possible. The corresponding inequalities for series are:

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2};\\ \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{\max\{m,n\}} < 4\left(\sum_{n=1}^{\infty}a_n^2\right)^{1/2}\left(\sum_{n=1}^{\infty}b_n^2\right)^{1/2},\end{gathered} \tag{1.4}$$

where the constant factors π and 4 are both the best possible.

Let H be a real separable Hilbert space, and $T : H\to H$ be a bounded self-adjoint semi-positive definite operator, then (see [4])

$$(x,Ty)^2 \le \frac{\|T\|^2}{2}\left[\|x\|^2\|y\|^2 + (x,y)^2\right], \tag{1.5}$$

where $x, y \in H$ and $\|x\| = \sqrt{(x,x)}$ is the norm of x.

Set $H = L^2(0,\infty) = \{f(x) : \int_0^\infty f^2(x)dx < \infty\}$ and define $T : L^2(0,\infty)\to L^2(0,\infty)$ as the following:

$$(Tf)(y) := \int_0^\infty \frac{1}{x+y}f(x)dx, \tag{1.6}$$

where $y \in (0,\infty)$. It is easy to see T is a bounded operator (see [5]). By (1.5), one has the sharper form of Hilbert's inequality as (see [4]),

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dx\,dy \le \frac{\pi}{\sqrt{2}}\left\{\int_0^\infty f^2(x)dx\int_0^\infty g^2(x)dx + \left(\int_0^\infty f(x)g(x)dx\right)^2\right\}^{1/2}. \tag{1.7}$$

Recently, Yang [6, 7] studied the Hilbert's inequalities by the norm of some Hilbert's type linear operators.

The main purpose of this article is to study the norm of a Hilbert's type linear operator with the kernel $A\min\{x,y\} + B\max\{x,y\}$ and give some new generalizations of Hilbert's inequality. As applications, we also consider some particular results.

2. Main results and applications

LEMMA 2.1. *Define the weight function* $\varpi(x)$ *as*

$$\varpi(x) := \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2} dy, \quad x\in(0,\infty),$$
$$\varpi(y) \triangleq \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2} dx, \quad y\in(0,\infty). \tag{2.1}$$

Then $\varpi(x)=\varpi(y)=D(A,B)$ *is a constant and* $0<D(A,B)<\infty$.

In particular, one has $D(1,1)=\pi$ *and* $D(1,0)=4$.

Proof. For fixed x, letting $t=y/x$, we get

$$\begin{aligned}
\varpi(x) &= \int_0^\infty \frac{1}{A\,\min\{x,y\}+B\,\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2} dy \\
&= \int_0^\infty \frac{1}{A\,\min\{1,t\}+B\,\max\{1,t\}}t^{-1/2}dt \\
&= \int_0^1 \frac{1}{At+B}t^{-1/2}dt+\int_1^\infty \frac{1}{A+Bt}t^{-1/2}dt \\
&= \frac{1}{\sqrt{AB}}\int_0^{A/B}\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+t}t^{-1/2}dt \\
&\le \frac{1}{\sqrt{AB}}\int_0^{\infty}\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_{0}^\infty \frac{1}{1+t}t^{-1/2}dt \\
&= \frac{2}{\sqrt{AB}}B\left(\frac{1}{2},\frac{1}{2}\right)<\infty.
\end{aligned} \tag{2.2}$$

therefore $0<D(A,B)<\infty$. Moreover,

$$\begin{aligned}
\varpi(y) &= \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2} dx \\
&= \int_0^\infty \frac{1}{A\min\{1,t\}+B\max\{1,t\}}t^{-1/2}dt \\
&= \int_0^1 \frac{1}{At+B}t^{-1/2}dt+\int_1^\infty \frac{1}{A+Bt}t^{-1/2}dt \\
&= \frac{1}{\sqrt{AB}}\frac{A^{-1+(1/2)}}{B^{1/2}}\int_0^{A/B}\frac{1}{1+t}t^{-1/2}dt+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+t}t^{-1/2}dt \\
&= \frac{1}{\sqrt{AB}}\int_0^{A/B}\frac{1}{1+u}u^{-1/2}du+\frac{1}{\sqrt{AB}}\int_{B/A}^\infty \frac{1}{1+u}u^{-1/2}du
\end{aligned} \tag{2.3}$$

(setting $t=1/u$).

Thus $\varpi(y) = D(A,B)$. In particular:

$$D(1,1) = \int_0^\infty \frac{1}{x+y}\left(\frac{y}{x}\right)^{1/2} dx = \int_0^\infty \frac{1}{1+t} t^{-1/2} dt = \pi,$$
$$D(0,1) = \int_0^\infty \frac{1}{\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2} dx = \int_0^\infty \frac{1}{\max\{1,t\}} t^{-1/2} dt = 4. \tag{2.4}$$

□

THEOREM 2.2. *Let $A \geq 0$, $B > 0$ and $T : L^2(0,\infty) \to L^2(0,\infty)$ is defined as follows:*

$$(Tf)(y) := \int_0^\infty \frac{1}{A\min\{x,y\} + B\max\{x,y\}} f(x)dx \quad (y \in (0,\infty)). \tag{2.5}$$

Then $\|T\| = D(A,B)$, and for any $f(x), g(x) \geq 0$, $f,g \in L^2(0,\infty)$, one has $(Tf,g) < D(A,B)\|f\|\|g\|$, that is,

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\} + B\max\{x,y\}} dx\, dy < D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)dx\right\}^{1/2}, \tag{2.6}$$

where the constant factor $D(A,B)$ is the best possible. In particular,

(i) *for $A = B = 1$, it reduces to Hardy-Hilbert's inequality:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx\, dy < \pi\left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)dx\right\}^{1/2}; \tag{2.7a}$$

(ii) *for $A = 0$, $B = 1$, it reduces to Hardy-Hilbert's type inequality:*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dx\, dy < 4\left\{\int_0^\infty f^2(x)dx\right\}^{1/2} \left\{\int_0^\infty g^2(x)dx\right\}^{1/2}. \tag{2.7b}$$

Proof. For $A > 0$, $B > 0$. Applying Hölder's inequality, we obtain

$$(Tf,g) = \left(\int_0^\infty \frac{f(x)}{A\,\min\{x,y\} + B\,\max\{x,y\}} dx, g(y)\right)$$
$$= \int_0^\infty \left(\int_0^\infty \frac{f(x)}{A\,\min\{x,y\} + B\,\max\{x,y\}} dx\right) g(y)dy$$

$$
\begin{aligned}
&= \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}\,dx\,dy \\
&= \int_0^\infty \int_0^\infty \frac{1}{A\min\{x,y\}+B\max\{x,y\}}\left[f(x)\left(\frac{x}{y}\right)^{1/4}\right]\left[g(y)\left(\frac{y}{x}\right)^{1/4}\right]dx\,dy \\
&\le \left\{\int_0^\infty \int_0^\infty \frac{f^2(x)}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{x}{y}\right)^{1/2}dx\,dy\right\}^{1/2} \\
&\quad\times\left\{\int_0^\infty \int_0^\infty \frac{g^2(y)}{A\min\{x,y\}+B\max\{x,y\}}\left(\frac{y}{x}\right)^{1/2}dx\,dy\right\}^{1/2} \\
&= \left\{\int_0^\infty \varpi(x)f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty \varpi(y)g^2(y)dy\right\}^{1/2} \\
&= D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)dy\right\}^{1/2} \\
&= D(A,B)\|f\|\|g\|.
\end{aligned}
\tag{2.8}
$$

Thus $\|T\| \le D(A,B)$ and the inequality (2.6) holds.

Assume that (2.8) takes the form of the equality, then there exist constants a and b, such that they are not both zero and (see [8])

$$af^2(x)\left(\frac{x}{y}\right)^{1/2} = bg^2(y)\left(\frac{y}{x}\right)^{1/2}. \tag{2.9}$$

Then, we have

$$af^2(x)x = bg^2(y)y \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.10}$$

Hence there exist a constant d, such that

$$af^2(x)x = bg^2(y)y = d \quad \text{a.e. on } (0,\infty)\times(0,\infty). \tag{2.11}$$

Without losing the generality, suppose $a \neq 0$, then we obtain $f^2(x) = d/(ax)$, a.e. on $(0,\infty)$, which contradicts the fact that $0 < \int_0^\infty f^2(x)dx < \infty$. Hence (2.8) takes the form of strict inequality, we obtain (2.6).

For $\varepsilon > 0$ sufficiently small, set $f_\varepsilon(x) = x^{(-1-\varepsilon)/2}$, for $x \in [1,\infty)$; $f_\varepsilon(x) = 0$, for $x \in (0,1)$. Then $g_\varepsilon(y) = y^{(-1-\varepsilon)/2}$, for $y \in [1,\infty)$; $g_\varepsilon(y) = 0$, for $y \in (0,1)$. Assume that the constant factor $D(A,B)$ in (2.6) is not the best possible, then there exist a positive real number K

with $K < D(A,B)$, such that (2.6) is valid by changing $D(A,B)$ to K. On one hand,

$$\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy < K\left\{\int_0^\infty f_\varepsilon^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g_\varepsilon^2(x)dx\right\}^{1/2} = K/\varepsilon. \tag{2.12}$$

On the other hand, setting $t = y/x$, we have

$$\begin{aligned}
&\int_0^\infty\int_0^\infty \frac{f_\varepsilon(x)g_\varepsilon(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\
&\quad= \int_1^\infty\int_1^\infty \frac{x^{(-1-\varepsilon)/2}y^{(-1-\varepsilon)/2}}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\
&\quad= \int_1^\infty x^{-1-\varepsilon}\int_{1/x}^\infty \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx\\
&\quad= \int_1^\infty x^{-1-\varepsilon}\int_0^\infty \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx\\
&\qquad- \int_1^\infty x^{-1-\varepsilon}\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx.
\end{aligned} \tag{2.13}$$

For $x \ge 1$, we get

$$\begin{aligned}
&\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\\
&\quad= \int_0^{1/x}\frac{t^{(-1-\varepsilon)/2}}{At+B}dt\\
&\quad\le \frac{1}{B}\int_0^{1/x} t^{(-1-\varepsilon)/2}dt\\
&\quad= \frac{1}{B}\frac{1}{1-(1+\varepsilon)/2}\left(\frac{1}{x}\right)^{1-(1+\varepsilon)/2}\\
&\quad\le \frac{4}{B}x^{-1/4}
\end{aligned} \tag{2.14}$$

(setting $0 < \varepsilon < 1/2$).

Thus

$$\begin{aligned}
0 &< \int_1^\infty x^{-1-\varepsilon}\int_0^{1/x} \frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx\\
&\le \frac{4}{B}\int_1^\infty x^{-1-\varepsilon-1/4}dx\\
&\le \frac{4}{B}\int_1^\infty x^{-1-1/4}dx = \frac{16}{B}.
\end{aligned} \tag{2.15}$$

Note that

$$\int_1^\infty x^{-1-\varepsilon}\int_0^{1/x}\frac{t^{(-1-\varepsilon)/2}}{A\min\{1,t\}+B\max\{1,t\}}dt\,dx=O(1). \tag{2.16}$$

So the inequality $\int_0^\infty\int_0^\infty(f_\varepsilon(x)g_\varepsilon(y)/(A\min\{x,y\}+B\max\{x,y\}))dxdy=(1/\varepsilon)[D(A,B)+o(1)]-O(1)=(1/\varepsilon)[D(A,B)+o(1)]$. Thus we get $(1/\varepsilon)[D(A,B,p)+o(1)]\le K/\varepsilon$, that is, $D(A,B)\le K$ when ε is sufficiently small, which contradicts the hypothesis. Hence the constant factor $D(A,B)$ in (2.6) is the best possible and $\|T\|=D(A,B)$. This completes the proof. □

Theorem 2.3. *Suppose that* $f\ge 0$, $A\ge 0$, $B>0$ *and* $0<\int_0^\infty f^2(x)dx<\infty$. *Then*

$$\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy<D^2(A,B)\int_0^\infty f^2(x)dx, \tag{2.17}$$

where the constant factor $D^2(A,B)$ *is the best possible. Inequality (2.17) is equivalent to (2.6).*

Proof. Let $g(y)=\int_0^\infty(f(x)/(A\min\{x,y\}+B\max\{x,y\}))dx$, then by (2.6), we get

$$\begin{aligned}0&<\int_0^\infty g^2(y)dy\\&=\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy\\&=\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\&\le D(A,B)\left\{\int_0^\infty f^2(x)dx\right\}^{1/2}\left\{\int_0^\infty g^2(y)dy\right\}^{1/2}.\end{aligned} \tag{2.18}$$

Hence, we obtain

$$0<\int_0^\infty g^2(y)dy=D^2(A,B)\int_0^\infty f^2(x)dx<\infty. \tag{2.19}$$

By (2.6), both (2.18) and (2.19) take the form of strict inequality, so we have (2.17). On the other hand, suppose that (2.17) is valid. By Hölder's inequality, we find

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy\\&\quad=\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]g(y)dy\\&\quad\le\left\{\int_0^\infty\left[\int_0^\infty\frac{f(x)}{A\min\{x,y\}+B\max\{x,y\}}dx\right]^2dy\right\}^{1/2}\left\{\int_0^\infty g^2(x)dx\right\}^{1/2}.\end{aligned} \tag{2.20}$$

By (2.17), we have (2.6). Thus (2.6) and (2.17) are equivalent.

If the constant $D^2(A,B)$ in (2.17) is not the best possible, by (2.20), we may get a contradiction that the constant factor in (2.6) is not the best possible. This completes the proof. □

It is easy to see that for $A=1$, $B=1$, the inequality (2.17) reduces to

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{x+y}dx\right]^2 dy < \pi^2 \int_0^\infty f^2(x)dx, \tag{2.21a}$$

and for $A=0$, $B=1$, the inequality (2.17) reduces to

$$\int_0^\infty \left[\int_0^\infty \frac{f(x)}{\max\{x,y\}}dx\right]^2 dy < 16 \int_0^\infty f^2(x)dx, \tag{2.21b}$$

where both the constant factors π^2 and 16 are the best possible.

3. The corresponding theorem for series

THEOREM 3.1. *Suppose that* $a_n, b_n \geq 0$, $A \geq 0$, $B > 0$, *and* $0 < \sum_{n=1}^\infty a_n^2 < \infty$, $0 < \sum_{n=1}^\infty b_n^2 < \infty$. *Then*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}} < D(A,B)\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}, \tag{3.1}$$

$$\sum_{n=1}^\infty \left[\sum_{m=1}^\infty \frac{a_m}{A\min\{m,n\}+B\max\{m,n\}}\right]^2 < D^2(A,B)\sum_{n=1}^\infty a_n^2, \tag{3.2}$$

where the constant factor $D(A,B)$ and $D^2(A,B)$ are both the best possible, (3.1) and (3.2) are equivalent. In particular,

(i) *for $A=1$, $B=1$, it reduces to Hardy-Hilbert's inequality:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{m+n} < \pi\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}; \tag{3.3a}$$

(ii) *for $A=0$, $B=1$, it reduces to Hardy-Hilbert's type inequality:*

$$\sum_{n=1}^\infty \sum_{m=1}^\infty \frac{a_m b_n}{\max\{m,n\}} < 4\left(\sum_{n=1}^\infty a_n^2\right)^{1/2}\left(\sum_{n=1}^\infty b_n^2\right)^{1/2}. \tag{3.3b}$$

Proof. Define the weight function $\omega(n)$ as

$$\omega(n) := \sum_{m=1}^\infty \frac{1}{A\min\{m,n\}+B\max\{m,n\}}\left(\frac{n}{m}\right)^{1/2}, \quad n \in N. \tag{3.4}$$

Then we obtain

$$\omega(n) < \varpi(n) = D(A,B). \tag{3.5}$$

Using the method similar to Theorem 2.2 and applying Hölder's inequality, we obtain

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}} \leq \left[\sum_{n=1}^{\infty}\omega(n)a_n^2\right]^{1/2}\left[\sum_{n=1}^{\infty}\omega(n)b_n^2\right]^{1/2}. \tag{3.6}$$

By (3.5), we obtain (3.1).

For $\varepsilon > 0$ sufficiently small, setting $\widetilde{a}_n = n^{-(1+\varepsilon)/2}$, $\widetilde{b}_n = n^{-(1+\varepsilon)/2}$, then

$$\begin{gathered}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{\widetilde{a}_m\widetilde{b}_n}{A\min\{m,n\}+B\max\{m,n\}} > \int_1^{\infty}\int_1^{\infty}\frac{f_\varepsilon(x)g_\varepsilon(y)}{A\min\{x,y\}+B\max\{x,y\}}dx\,dy,\\ \left\{\sum_{n=1}^{\infty}\widetilde{a}_n^2\right\}^{1/2}\left\{\sum_{n=1}^{\infty}\widetilde{b}_n^2\right\}^{1/2} = \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} < 1+\int_1^{\infty}\frac{1}{t^{1+\varepsilon}} = 1+\frac{1}{\varepsilon}.\end{gathered} \tag{3.7}$$

If the constant factor $D(A,B)$ in (3.1) is not the best possible, then applying the result of Theorem 2.2, we can get the contradiction. Let $b_n = \sum_{m=1}^{\infty}(a_m/(A\min\{m,n\}+B\max\{m,n\}))$ and we can obtain the following relation:

$$\begin{aligned}&\sum_{n=1}^{\infty}\left[\sum_{m=1}^{\infty}\frac{a_m}{A\min\{m,n\}+B\max\{m,n\}}\right]^2\\ &\quad= \sum_{n=1}^{\infty}b_n^2 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{A\min\{m,n\}+B\max\{m,n\}}.\end{aligned} \tag{3.8}$$

Applying (3.1) and the method similar to Theorem 2.3, we get (3.2), and (3.2) is equivalent to (3.1) with the best constant. □

Acknowledgments

The work was partially supported by the Emphases Natural Science Foundation of Guangdong Institution of Higher Learning, College and University (no. 05Z026). The authors would like to thank the anonymous referee for his or her suggestions and corrections.

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